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TCHEBYCHEFF INEQUALITIES AS A BASIS FOR STATISTICAL TESTS

C. D. Smith

Introduction

One of the principle uses of mathematical statistics is to furnish a basis for testing the value of statistical results obtained from samples. The main purpose of this paper is to present the essential steps in the development of mathematical tests in the approximate order of their appearance. We shall see how the original theorem of Tchebycheff demonstrates the basic principle of the probability test. Some further developments of the theory are given in the last section. Beginning in the seventeenth century we find published papers from many sources giving different test results. Presentation of the principle test results so as to bring the theory up to date should be of some use to those interested in present practices and further developments.

The Original Problem

One may ask for the extent to which sample observations characterize the population from which the samples are drawn. The mathematical theory of probability began with the publication of Ars Conjectandi by James Bernoulli[1] 1713, published eight years after his death. To illustrate the original concept we assume a variable u which takes only the integral values from 1 to 20 and may on some one occasion have a value greater than 15. The probability of the occurrence is 5/20 or 1/4 since there are five possible values greater than 15. If the 20 values of u are observations taken from a larger population source, the ratio 1/4 is called the relative frequency of values of u greater than 15. The extent to which a relative frequency approximates the corresponding probability in the universe from which the sample is drawn constitutes the basis for statistical tests.

Following this beginning the theory languished for the period of a century being considered spasmodically by, DeMoivre, Sterling, Maclaurin, Bayes, Euler, and Gauss. In 1812, the publication of the Theorié Analytiqué, by Laplace, brought the theory up to date. During the remainder of the 19th century important contributions were made by, Cauchy, Gauss, Bienayme, Tchebycheff, Laxis, Charlier, Keynes, and Karl Pearson. Since 1900 the works of Karl Pearson and R. A. Fisher have pointed the way for the many advances which have been made in the theory and application of statistical tests by means of samples.

The problem of testing the behavior of a variable by sample observations includes the manner of occurrence over the range of values and the probability of the occurrence of certain assigned values of the variable. It seems that the sampling distribution was first considered up to the time of approximately 1850.

The Binomial and the Normal Distribution

The binomial distribution may be written in the form $N(p+q)^n$, where N is the number of trials, p is the probability of success in one trial, and q is the probability of failure. The number of categories in which the event falls is determined by n. As an easy illustration take the case of two coins thrown 100 times for the distribution of heads and tails. Since $p = q = .5$, we have $100(.5+.5)^2 = 25+50+25$, a symmetrical distribution about the middle value. For the probability distribution of $(p+q)^n$ the average, or expected value, of u is $u = np$, and the standard deviation, or average range of values is $\sigma = \sqrt{npq}$. For $p \neq q$ the distribution is non-symmetric, or skewed.

The problem of finding a continuous function, $y = \phi(x)$ to represent the symmetric distribution of a variable gave the normal distribution function,

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

The binomial function is represented geometrically by a polygon symmetric about the middle value. The corresponding normal function is represented by a curve symmetric to its most probable value. In the normal function we have the first criterion for testing the expected behavior of a variable. The question to be answered is, how close do samples of observed values conform to the corresponding expected values of the variable? A test of the probability that the sample was drawn from a normal population would be a basis for judging the value of evidence obtained from samples. The Chi-square test which serves this purpose is due to Karl Pearson.

The Chi-Square (χ^2) Distribution

Karl Pearson[13] proposed a test in the form $\chi^2 = \Sigma(f_i - f)^2/f$, where f_i is the frequency of observed values corresponding to the frequency f of the expected values. It was necesssry to determine the distribution of χ^2 so that the probability of a given value of χ^2 could be computed. The distribution function was found to be

$$T_{K-1}(\chi^2)d(\chi^2) = \frac{(\chi^2)^{\frac{K-3}{2}} e^{-\frac{1}{2}\chi^2}}{2^{\frac{K-1}{2}} \Gamma(\frac{K-1}{2})} d\chi^2,$$

where K-1 is the number of degrees of freedom of the variable. The probability for a given value χ_0^2 is caluclated from the integral

$$P(\chi^2 > \chi_0^2) = \int_{\chi_0^2}^{\infty} T_{k-1}(\chi^2) d(\chi^2).$$

Tables of values of χ^2 have been calculated for certain values of the probability P. By calculating χ^2 for an observed sample and given population we obtain evidence of the goodness of fit of the sample to the population. There are two points of view in the interpretation of results which were discussed by Pearson and Fisher. First we may assume a given population and test a sample to see whether or not it is bad, or second we assume that a sample is a good sample and test a population to determine whether or not it may properly represent the population from which the sample is drawn.

Another approach to probability criteria for statistical tests began in 1853 with a paper by Bienayme[3] in Comptes Rendus. He followed the work of Gauss whereby a variable is characterized by its standard deviation as a measure of dispersion about the centroidal (average) value.

The Tchebycheff Inequality

The first theorem which sets a bound for the probability that a variable will depart from its expected value by more than a given amount appeared in 1867 in a paper by Tchebycheff[4] The paper was published in the Journal of Mathematics by Liouvelle. The theorem is stated as follows;

If we represent by a, b, c,$\cdots$ the mathematical expectations of quantities x, y, z,$\cdots$, and by a_1, b_1, c_1,$\cdots$the mathematical expectations of the squares x^2, y^2, z^2, $\cdots$the probability that the sum $x + y + z \cdots$ is contained between the limits

$$a + b + c + \cdots \pm \lambda\sqrt{a_1 + b_1 + c_1 \cdots - a^2 - b^2 - c^2 \cdots}$$

is greater than $1 - 1/\lambda^2$ for $\lambda \geq 1$. For one variable the theorem may be more conveniently stated in the following form.

If u is the deviation of a variable from its expected value, the probability P that u is numerically greater than $\lambda\sigma$ is bounded by the inequality $P \leq 1/\lambda^2$ for $\lambda \geq 1$.

The theorem is remarkable in that no assumption is made regarding the nature of the distribution. We have here a test of the occurrence of u regardless of the manner of its variation. Beginning with this basic test we shall see how a variety of tests have developed which set the probability level for the occurrence of an observed value of a variable. Experiments with known populations have shown that cases arise where the bound is not sufficiently close to P for practical application. The next step to develop a closer inequality was made by Karl Pearson[5] in 1919. By using even moments of higher order the generalized inequality takes the form $P_{\lambda\sigma} \geq M_{2n}/\lambda^{2n} M_2^n$, where M is a moment about the mean of the distribution. The estimate of P is improved by using moments of higher order than the second.

B. H. Camp[6] reduced the estimate of P still further in 1922 by restricting the distribution to a monotonic decreasing function of $|u| \geq c\sigma$, $c \geq 0$. With u measured from the mean as origin the generalized inequality is

$$P_{\lambda\sigma} \leq \frac{\beta_{2n-2}}{\lambda^{2n}(1+1/2n)^{2n}} \cdot \frac{1}{1+\phi} + \theta, \theta = \frac{\phi P_{c\sigma}}{1+\phi}, \text{ and } \phi = \frac{[2cn/(2n+1)]^{2n}}{(2n+1)(\lambda/c-1)}$$

Here the bound for a given value of u is approximately 50 per cent of the bound obtained by Pearson.

During the same year M. B. Meidell[7] obtained a bound analogous to that of Camp by using two further considerations. First assume that deviations are measured from the most probable value of u, and second that an assigned value $d = \lambda(M_n)^{1/n}$, where M_n is the momentof any order about the most probable value. The resulting inequality is

$$P_d \leq \frac{1}{\lambda^n(1+1/n)^n}.$$

In 1923 a paper by Narumi[8] appeared in Biometrika which gives two further generalizations. First beginning with an arbitrary origin the corresponding problem for positive deviations is given by the transformation $\int_{-\infty}^{\infty}(f(u)+f(-u)]\,du=1$. The resulting inequality is analoggous to that of Meidell. He also obtained the inequality for an increasing function with deviations from an arbitrary origin.

Some further results are given in a paper by the writer[9] which appeared in the American Journal of Mathematics, January 1930. In this paper the original inequality of Tchebycheff is generalized in the following manner. By measuring deviations of a variable from an arbitrary origin we may fold over the negative deviations to the positive side and consider a set of positive deviations u_i, $i=1$ to n and an arbitrary value d such that $u_c < d \leq u_{c+1}$. If u_i has the probability p_i, the probability P_d that u is greater than d is

$$P_d = \sum_{i=c+1}^{n} p_i \cdot M_n = \sum_{i=1}^{c} u_i^n p_i + \sum_{i=c+1}^{n} u_i^n p_i \geq d^n \sum_{i=c+1}^{n} p_i = d^n P_d \cdot \therefore P_d \leq M_n/d^n .$$

Obviously this inequality can be improved if we reduce the numerator by any amount not greater than the loss of $\sum_{i=1}^{c} u_i^n p_i$, and the loss by substituting d for values of $u_i > d$.

To obtain some compensation for this loss we assume a function $f(x)$ for which $y = P_x$ is the probability function. Let $f(x)$ be a decreasing function of x so that the curve of $y = P_x$ is concave upward. By integrating under a tangent to $y = P_x$ we obtain a moment less than M_n. By setting $d = \lambda(M_n)^{1/n}$ and finding the tangent which minimizes the difference between the two moments we obtain the inequality $P_d \leq 1/\lambda^n(1+1/n)^n$, analogous to that of Meidell. Note that this method proves that we have the best inequality obtainable from a tangent to the curve of $y = P_x$.

Next let $f(x)$ increase from $x=0$ to $x=c\sigma$ and decrease beyond $x=c\sigma$. The curve for $y=P_x$ is now concave downward for $x < c\sigma$ and concave upward for $x > c\sigma$. By calculating the moment under the chord from $x=0$ to $x=c\sigma$ and the moment under the tangent for the area beyond $x=c\sigma$ the result is

$$P_{\lambda\sigma} \leq [\beta_{2r-2} - c^{2r}/(2r+1)] \;/[(\tfrac{\lambda}{\theta})^{2r} - c^{2r}/(2r+1)] .$$

Here θ is determined from the equation

$$\lambda = [2r/(2r+1)]\,[\lambda^{2r+1} - (c\theta)^{2r+1}]\;/\theta[\lambda^{2r} - (c\theta)^{2r}].$$

A paper by the writer[10] in the Annals of Mathematical Statistics, 1939, reduces the above result for cases where the range of the distribution terminates. The closer inequality is

$$P_{\lambda\sigma} \leq \frac{\beta_{2r-2} - c^{2r}\left\{1 - \dfrac{2rc}{\lambda(2r+1)}\right\}}{(\frac{\lambda}{\theta})^{2r} - c^{2r}\left\{1 - \dfrac{2rc}{\lambda(2r+1)}\right\}}$$

Beginning about the year 1895 another approach to the problem of statistical tests was made by the process of curve fitting. The idea was to determine a curve to represent the population from which a sample is drawn and then calculate the probabilities from the curve. The method leaves open the question as to whether the curve actually describes the correct population. A brief discussion of the method will illustrate the second phase in methods for testing statistics.

Gauss demonstrated the value of the Normal Curve and other curves obtained by the method of least squares for characterization of observations obtained by experiment. Pearson found that statistics do not conform to the normal law in many cases and that some more adequate method is needed. In a series of papers[13] during the years 1895-1916 he developed the Pearson System of Frequency Curves. Beginning with the general solution of the differential equation $dy/dt = (m-2)y/(a+bt+ct^2)$ the particular solutions for special cases give a variety of different curve types. The choice of a suitable curve is made by comparison with the sample mean, sigma, and skewness. The curve is used for calculating probability estimates of sample values. The normal curve is one of the curves in the Pearson system.

Another method of characterizing samples is known as the Gram-Charlier System which begins with the series

$$F(x) = C_0 f(x) + C_1 f^{(1)}(x) + \cdots + C_n f^{(n)}(x) + \cdots .$$

The function $f(x)$ is called the generating function and $f^{(n)}(x)$ is the nth derivative of $f(x)$. Two types have been developed from this series one of which uses the normal function as generating function. The equations for the Pearson system and also for the Gram-Charlier system with illustrative applications appear in the literature. By comparison with the original method of Tchebycheff we see that the approach by curve fitting requires judgment of the closeness of the fit to the sample values, and the other requires judgment of the closeness of the Tchebycheff bound.

The present phase of statistical testing began with problems which arose in the theory of small samples. A solution to the problem of small sample tests appears in the works of Student and R. A. Fisher. Many others have contributed to the theory during the past thirty years. The results obtained by Student and Fisher are given in the next section.

Statistical Tests for Small Samples

In 1908 a paper by Student[15] on the, Probable Error of the Mean, appeared in Biometrika. The original methods tested the sample as a whole, the Tchebycheff method tested given values of the variable, and now we have the propossl by Student to test sample averages. Due to the variability of the standard deviation s in small samples he used the test, $t = (\bar{x}-\tilde{x})(n-1)^{1/2}/s$, where t is not normally distributed for N small. To calculate the probability for values of t he found the distribution of $z = (\bar{x}-\tilde{x})/s$ which took the form

$$F(Z) = \frac{\Gamma(n/2)}{\pi^{\frac{1}{2}}\Gamma(\frac{n-1}{2})} (1+Z^2)^{-n/2}$$

Here $\bar{x}$ is the sample mean, $\tilde{x}$ is the population mean, and s is the estimate of the standard deviation. For a given sample average $\bar{x}$ the probability is calculated from $F(z)$ to test the departure of the sample average from an expected value representing a population average. This marks the first notable advance in the test of averages taken from small samples.

$$F_n(t) = K_n(1 + t^2/n)^{-(n+1)/2}$$

where K_n is s constant determined by $n = N-1$. A table giving values of t for certain assigned values of P and n serves as a basis for applying the test. Fisher also solved the problem of testing the difference between two variances u^2 and v^2. Beginning with the statistic $z = \frac{1}{2}(\log_e u^2 - \log_e v^2)$ he developed the distribution function $G(z)$ from which the probabilities are calculated for values of z. Tables for n_1 and n_2 of the samples and assigned values of P are used to make the variance test as evidence of the difference between samples. Snedecor transformed Fisher's $z = \frac{1}{2}(\log_e u^2 - \log_e v^2)$ into the more convenient form $F = u^2/v^2$ and prepared the F-table for testing the ratio between the variances u^2, v^2.

For large samples the statistic t is approximately normally distributed with a value near 2 for $P = .05$, and near 3 for $P = .01$. Fisher suggested that the 5 per cent and 1 per cent levels of P are adequate for judging the order of significance in tests with z snd F. These methods have lead to some new procedures for tests based on small samples.

Testing Procedures for Small Samples

Prior to the work of Student and Fisher, Karl Pearson used the ratio between variances as basis for measuring correlation between variables. When a line is fitted to values of y for corresponding values of x the line has the property that the sum of the squares of deviations of y-points from the line plus the sum of the squares of deviations of the points on the line from the mean of the y-points equals the sum of the squares of deviations of the y-points from the mean. Representing the averages of the squares in order by S_y^2, S_e^2, and σ_y^2, we have the relation

$$S_y^2 + S_e^2 = \sigma_y^2, \quad \text{or} \quad S_y^2 = \sigma_y^2(1 - S_e^2/\sigma_y^2).$$

Pearson definded r in the ratio $r^2 = S_e^2/\sigma_y^2$ as the measure of correlation between x and y. Obviously r^2 is less than or equal to 1.

Fisher proposed a test procedure where the ratio between variances is greater than 1, and the F-test applies. The method is known as the Analysis of Variance where observed values of a variable are first arranged in strata according to a preassigned plan. Calculate the variances within strata of each item from its stratum mean and the variance of the stratum means from the general mean. The F-ratio is used to test the significance of variation between the strata. The method has a large place in the field of small sample tests because of the many ways of determining strata by sampling techniques.

A more recent procedure is the method of Sequential Analysis developed by Abraham Wald during World War II for use by the armed forces. His book entitled, Sequential Analysis, explains the original plan. A simple case to illustrate the procedure arises when we need to test for acceptance in a population of items where each item may be classified as acceptable or not acceptable. First a probability level is set at which the number of defects in the population would be acceptable. There are two associated probabilities, first that a sample may accept a bad lot, and second that a sample may reject a good lot. From these conditiins Wald found a linear equation for the number of defects acceptable and the equation for the number not acceptable as the sample number n increases. A table for acceptance and rejection may be prepared from the equations. Items are then drawn from the population and the number of defects noted as the sample number increases. The cumulative number of defects in the sample will ultimately become less than the acceptance number or greater than the rejection number, at which time the sample is terminated with the decision based on the result. In case of a very bad lot or a very good lot the decision is reached quickly. When application of the sequential sample is appropriate it effects a considerable saving in time as compared to other procedures for testing samples.

Tchebycheff Limits for Problems in More than One Variable

The inequality for two variables, x and y, is developed by Berge[16] in Biometrika, 1938. With $\bar{x}=\bar{y}=0$, standard deviations σ_x and σ_y, and correlation r between x and y, a rectangle is bounded by the lines $x=\pm\lambda\sigma_x$, and $y=\pm\lambda\sigma_y$. The probability that a point (x,y) falls inside of the rectangle is bounded by the inequality

$$P \geq 1 - \frac{1+\sqrt{1-r^2}}{\lambda^2}$$

The formuls becomes $P \geq 1-2/\lambda^2$ when the variables are not correlated.

A paper in The Annals of Mathematical Statistics, March 1947, by Birnbaum, Raymond, and Zuckerman,[11] gave the following result for independent random variables X and Y with expectations X_0 and Y_0, and variances σ_x^2 and σ_y^2. For any $s>0, t>0$, such that $\frac{\sigma_x^2}{s^2} \leq \frac{\sigma_y^2}{t^2}$ the result is

$$P\left[\frac{(x-x_0)^2}{s^2}+\frac{(y-y_0)^2}{t^2} \geq 1\right] \leq L(s,t) \leq 1 .$$

The value of $L(s,t)$ depends on the relation $\frac{\sigma_x^2}{s^2}+\frac{\sigma_y^2}{t^2} \lesseqgtr 1$. Conditions are given for which the equality holds and a special case is given for more than two variables.

In The Annals of Mathematical Statistics, 1948, B. H. Camp[12] extended his inequality for one variable to space of n-dimensions. The extension is based on the definition of a contour moment which may be described in three dimensions as follows. Let $y=f(t_1,t_2)$ represent a surface determined by points (t_1, t_2) of the base plane. For an assigned value of y the points (t_1, t_2) determine a contour on the surface with area x. Considering y as a function of the contour area the contour moment from $0 \leq x \leq x_0$ is given by the integral $\widehat{\mu}_{2r}=\int_0^{x_0} x^r y \, dx$. defining next an n-space in which y is a monotonic decreasing function of x the inequality in terms of contour moments is

$$P_{\lambda\widehat{\sigma}} \leq \frac{\widehat{\alpha}_{2r}}{\lambda^{2r}(1+1/2r)^{2r}}, \quad \widehat{\alpha}_{2r} = \widehat{\mu}_{2r}/\widehat{\sigma}^{2r}.$$

Tables are given to aid in making calculations. A special case is given where the equality is realized.

Some Further Developments

We obtain some additional inequalities for cases of two variables by proceding in a different manner from the preceding developments. Let u_1 and u_2 be independent variables with positive values represented by points (u_1, u_2) of the plane. We know that the probability P_1 that $u_1 > \lambda(M_{1,n})^{1/n}$ is bounded by $P_1 \leq 1/\lambda^n$, and the probability P_2 that $u_2 > (M_{2,n})^{1/n}$ is bounded by $P_2 \leq 1/\lambda^n$. Let P be the probability that a point (u_1, u_2) falls outside of the rectangle formed by $\lambda(M_{1,n})^{1/n}$ and $\lambda(M_{2,n})^{1/n}$. Since the variables are independent and either deviation places the point outside of the rectangle we have $P = P_1 + P_2$. The resulting inequality is $P \leq 2/\lambda^n$. This result generalizes the enequality of Berge in the case where $r = 0$. In a similar manner for three variables the probability that a point falls outside of a rectangular solid determined by the dimensions $\lambda(M_{1,n})^{1/n}$, $\lambda(M_{2,n})^{1/n}$, $\lambda(M_{3,n})^{1/n}$ is bounded by $P \leq 3/\lambda^n$. Extension to space of n-dimensions gives $P \leq n/\lambda^n$.

Next let u_1 and u_2 be correlated with r the coefficient of correlation. If x and y are deviations from the respective means, the regression equation of y on x is $y = r(\sigma_y/\sigma_x)x$. The second moment S_y^2 of the deviations of points from the regression line is $S_y^2 = \sigma_y^2(1 - r^2)$. Therefore, the probability that a point deviates from the regression line by an amount greater than d is bounded by $P_d \leq \sigma_y^2(1 - r^2)/d^2$, or $P \leq (1 - r^2)/\lambda^2$, $d = \lambda\sigma$. Extension of the inequality to the fourth moment requires considerable calculations A form of the result turned out to be

$$S_y^4 = \beta_{2,y}(1 - 4r^2) + 6r^2(1 - r^2 + \frac{r^2}{2})\,\beta_{2;4}, \quad \beta_2 = \mu_4/\sigma^4$$

The inequality $P_d \leq S_y^4/d^4$ gives a closer bound than that obtained by the second moment.

A boundary value based on contours of equal probability may be obtained from the following considerations. The Normal frequency surface is represented by the function

$$Z = \frac{1}{2\pi\sigma_x\sigma_y(1 - r^2)^{1/2}}\, e^{-\frac{1}{2(1-r^2)}\left[\frac{x^2}{\sigma_x^2} + \frac{x^2}{\sigma_y^2} - \frac{2r+y}{\sigma_x\sigma_y}\right]}$$

In 1912, H. L. Rietz[17] discussed the surface in the form

$$Z = \frac{1}{2\pi\sigma_x\sigma_y(1 - r^2)^{1/2}}\, e^{-\frac{\lambda^2}{2(1-r^2)}}$$

where λ is a parameter which determines the size of the ellipse of equal probabilities,

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2rxy}{\sigma_x\sigma_y} = \lambda^2$$

The ellipse is the projection on the x,y-plane of points of equal probability on the surface. The area of the ellipse is $A = \pi\lambda^2\sigma_x\sigma_y/(1 - r^2)^{1/2}$. Moments of volume under the surface may be represented as functions of λ in the form

$$M_{2s} = \int_0^\infty e^{-\lambda^2/2(1-r^2)} \cdot \lambda^{2s+1}/(1 - r^2)\, d\lambda = [2s(2s-2)\cdots 2](1 - r^2)^s.$$

We may now ask for the probability that the projection of a point of equal probability will fall outside of the ellipse. Since z is a decreasing function, the result is

$$P_\lambda \le \frac{[2s(2s-2)\cdots 2]\,(1-r^2)^s}{\lambda^{2s}(1+1/2s)^{2s}}$$

In the special case where $\sigma_x = \sigma_y$ and $r = 0$ the ellipse becomes a circle with radius λ. In this case the probability that a point of projection falls outside of the circle with radius λ is bounded by the inequality

$$P_\lambda \le \frac{2s(2s-2)\cdots 2}{\lambda^{2s}(1+1/2s)^{2s}}$$

To summarize the theory of statistical tests we begin with the binomial and the normal distributions. Pearson's Chi-square test for the goodness of fit of a sample followed. Probability bounds for the occurrence of assigned values of the variable began with the theorem of Tchebycheff. The method of curve fitting was introduced by the Pearson system and the Gram-Charlier system of frequency curves. Special test procedures for small samples and extension of bounds to more than one variable bring the theory us to date.

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A NOTE ON COLOGARITHMS

Sam Selby

Recent publications of elementary texts in Algebra and Trigonometry stress the importance of scientific notation for numbers as it relates to the convenience of finding the characteristics for the common logarithms of numbers. This follows from the fact that if $N = P \cdot 10^k$, where $1 \leq P < 10$, $\log N = \log P + k \log 10 = \log P + k$, and of course the characteristic is k, the $\log P$ being in toto the mantissa, and contained directly in regular common logarithmic tables.

The writer of this note wishes to point out that an analogous situation exists for characteristics of the common cologarithms of numbers. This follows from the fact that if $N = R \cdot 10^m$, where $0.1 \leq R < 1$, then $\text{colog } N = \text{colog } R + m \text{ colog } 10 = \text{colog } R - m$, and $-m$ is the characteristic for colog N. For example, the characteristics for colog 0.000015 and colog 57.2 would respectively be 4, and -2, since $0.000015 = 0.15 \cdot 10^{-4}$ and $57.2 = 0.572 \cdot 10^2$.

If a table of common logarithms is available which gives directly the negative decimal values for the range included for R, it becomes quite simple to find the common cologarithm of any particular number. Thus if log 0.15 and log 0.572 were contained directly in a table as -0.8239 and -0.2434, it would follow that colog 0.000015 = 4.8239 and colog 57.2 = 0.2434 - 2 = 8.2434 - 10.

It may be noted that such a four-place common logarithm table of the type mentioned herein is to be found either in the Mathematical Tables from the Handbook of Chemistry and Physics, pages 16 and 17, or its recent successor, the tenth edition of the Standard Mathematical Tables, pages 18 and 19.

When teaching cologarithms of numbers, the experience of the writer has been that it is easier to present and easier for the students to understand this topic when using this rule for characteristics and the logarithm table mentioned herein.

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Note on "A Note On Cologarithms": Whenever we set up a certain amount of machinery to facilitate a procedure, we better be sure that the procedure is quantitatively sufficiently productive to justify the machinery set up. This applies to the use of cologs as well as, say, to the production of automobiles. Editor

A REMARK ON THE QUOTIENT LAW OF TENSORS

T. K. Pan

The quotient law of tensors is one of several important devices used for the determination of the tensor character of a system. Since the concept involved in the law is quite elementary and its proof is already taken care of with sufficient detail and clarity in English language books on the subject, there seems no need of further discussion. But in applying the law, many students apparently misunderstand it, making several serious mistakes which will be illustrated by familiar examples found in most books. As some of them even offer reasons in support of their mistakes from their own reference books[1], it seems necessary to have the statement together with a proof of a very simple and obvious theorem.

This remark depicts separately some assumptions which are involved explicitly or implicitly in the quotient law of tensors and which most students are apt to overlook. It is the hope of the author that this illustration might be of some help in clarifying misunderstanding about this law.

The quotient law of tensors is given in different forms by different authors; but it can be readily shown that they are essentially equivalent, no matter whether they are so named or not. For convenience, we take the one given by L. P. Eisenhart as illustration [*An Introduction to Differential Geometry*, Princeton Press, 1949, p. 97]. It reads as follows:

If a set of functions

$$b^{r_1 \cdots\cdots r_m}_{p_1 \cdots\cdots p_n} \text{ and } b'^{s_1 \cdots\cdots s_m}_{q_1 \cdots\cdots q_n}$$

of x^i and x'^i respectively are such that

$$b^{r_1 \qquad r_m}_{p_1 \cdots p_h \cdots p_n} \lambda^{p_h} \text{ and } b'^{s_1 \cdots\cdots s_m}_{q_1 \cdots q_h \cdots q_n} \lambda'^{q_h}$$

for any p_h and q_h are components of a tensor, where λ^i and λ'^i are components of an arbitrary vector in these respective coordinates, then the given functions are components of a tensor.

There are three assumptions involved in the above statement. They are:

A_1. The set of functions is independent of the vector λ^i.

A_2. The vector λ^i is arbitrary.

A_3. In the product there is contraction, that is, the product is an inner product.

[1]Such as H. Lass' *Vector and Tensor Analysis* [McGraw-Hill Book Co., 1950, page 276].

Replacing the arbitrary vector λ^i by an arbitrary tensor or by a product of *distinct* arbitrary vectors gives other forms of the quotient law.

Suppose A_3 is observed. Neglect of either A_1 or A_2 will lead to a wrong conclusion. For example, let λ^i be an arbitrary vector and $a_{ij}\lambda^i\lambda^j$ be an invariant. It is obvious that a_{ij} is not necessarily a tensor and is a tensor only under further restrictions, such as a symmetric tensor when $a_{ij} = a_{ji}$. But the following proofs are found not infrequently in works of some students: (1) Since λ^j is arbitrary, by the quotient law $a_{ij}\lambda^i$ is a tensor. Repetition of the same argument leads to the conclusion that a_{ij} is a covariant tensor of the second order. (2) Since λ^i is an arbitrary vector, $\lambda^i\lambda^j$ is a tensor. Hence the quotient law gives a_{ij} as a tensor. The first mistake is due to overlooking A_1, since $a_{ij}\lambda^i$ is dependent on the vector λ^i; while the second mistake is due to overlooking A_2 since $\lambda^i\lambda^j$ is not an arbitrary tensor.

Suppose A_3 is not observed. There is no contraction in the product. Then the assumption of arbitrariness in A_2 is obviously not needed. For example, let $B^p_{rst}\lambda^q$ be a tensor of order five. Let λ^q be a vector. Then

$$\left(B^p_{rst} - B'^{i}_{klm}\frac{\partial x^b}{\partial x'^i}\frac{\partial x'^k}{\partial x^r}\frac{\partial x'^l}{\partial x^s}\frac{\partial x'^m}{\partial x^t}\right)\lambda^q = 0.$$

Any nonzero component of λ^q will reduce the above equations into

$$B^p_{rst} = B'^{i}_{klm}\frac{\partial x^p}{\partial x'^i}\frac{\partial x'^k}{\partial x^r}\frac{\partial x'^l}{\partial x^s}\frac{\partial x'^m}{\partial x^t}$$

Hence B^p_{rst} is a tensor if λ^q is not a zero vector. By a similar argument we have:

THEOREM. *If the product of a set of functions and a non-zero tensor is a tensor, then the given functions are components of a tensor.*

COROLLARY. *If the product of a set of functions and a scalar different from zero is a tensor, then the given functions define a tensor of the same kind.*

Either one of the above two statements is not what is usually called the quotient law of tensors. Moreover, it is misleading to apply the quotient law of tensors for the determination of tensor character of a system when there is no contraction.

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OPERATOR MATHEMATICS II

Jerome Hines

5. This paper is a continuation of "Foundations of Operator Mathematics" ibid: May-June, 1952. The aims of this paper include those of the preceding paper in which an algebraic background was developed for operators and then a calculus of operators. Finally an infinite series expansion of all operators, linear and non-linear, possessing an infinite number of derivatives was given. In this paper a generalization of l'Hospital's Rule is made to include indeterminate forms in operators. Next, a definition is made for the logarithm of an operator, and a study is made of the logarithm of the derivative. The results obtained from the logarithm of the derivative will be found to be interestingly different from those obtained by Volterra by the methods of integral equations. We will also find the Volterra operator to be inconsistent with the rules of operator equations developed in the previous paper.

6. The Extended l'Hospital's Rule:

Since this paper is a continuation of the paper referred to above, we will continue in the number series of the first paper. Thus all numbers up to and including section 4 are part of the first paper.

From equation 4.11 an arbitrary operator, A, possessing an infinite number of derivatives can be expressed by the infinite series:

4.11 $$A \overset{S}{=} \sum_{i=0}^{\infty} \frac{(x-x_0)^i \; {}_{x_0}D^i \cdot A}{i!}$$

wherein A contains the independent variable x. Likewise assume B expandable about x_0,

$$B \overset{S}{=} \sum_{j=0}^{\infty} \frac{(x-x_0)^j \; {}_{x_0}D^j \cdot B}{j!}$$

Let $A \overset{S}{=} \Omega$, $B \overset{S}{=} \Omega$, at $x = x_0$, where, from the previous paper Ω is defined by

$$\Omega a \equiv 0 \quad (\text{for all } a\text{'s})$$

Then $B^{-1}A$ at $x = x_0$ will be called indeterminate, parallel to the conventions of function theory. Now note that

$$B^{-1}A \overset{S}{=} \left[\sum_{j=0}^{\infty} \frac{(x-x_0)^j \; {}_{x_0}D^j \cdot B}{j!} \right]^{-1} \left[\sum_{i=0}^{\infty} \frac{(x-x_0)^i \; {}_{x_0}D^i \cdot A}{i!} \right]$$

But at $x = x_0$, A and B are equal to the null operator, Ω, whence, dropping first terms and using equation 1.9

$$B^{-1}A \overset{S}{=} \left[\sum_{j=1}^{\infty} \frac{(x-x_0)^j \, {}_{x_0}D^j \cdot B}{j!}\right]^{-1} \left[\sum_{i=1}^{\infty} \frac{(x-x_0)^i \, {}_{x_0}D^i \cdot A}{i!}\right]$$

$$\overset{S}{=} \left\{ [x-x_0] \left[\sum_{j=1}^{\infty} \frac{(x-x_0)^{j-1} \, {}_{x_0}D^j \cdot B}{j!}\right] \right\}^{-1} [x-x_0] \left[\sum_{i=1}^{\infty} \frac{(x-x_0)^{i-1} \, {}_{x_0}D^i \cdot A}{i!}\right]$$

$$\overset{S}{=} \left[\sum_{j=1}^{\infty} \frac{(x-x_0)^{j-1} \, {}_{x_0}D^j \cdot B}{j!}\right]^{-1} \frac{1}{x-x_0} \cdot (x-x_0) \left[\sum_{i=1}^{\infty} \frac{(x-x_0)^{i-1} \, {}_{x_0}D^i \cdot A}{i!}\right]$$

$$\overset{S}{=} \left[\sum_{j=1}^{\infty} \frac{(x-x_0)^{j-1} \, {}_{x_0}D^j \cdot B}{j!}\right]^{-1} \left[\sum_{i=1}^{\infty} \frac{(x-x_0)^{i-1} \, {}_{x_0}D^i \cdot A}{i!}\right]$$

If we evaluate this expression for $B^{-1}A$ at $x = x_0$, all terms in both the summations become zero with the exception of $i = 1$, $j = 1$, e.g.

6.1 $$B^{-1}A \overset{S}{=} (D \cdot B)^{-1} D \cdot A$$

at $x = x_0$, if $A \overset{S}{=} B \overset{S}{=} \Omega$ at x_0.

As an example of the Extended l'Hospital's Rule let us apply 6.1 to equation 3.8 (for the derivative of an operator) from the previous paper:

3.8* $$D \cdot F \overset{S}{=} DF - {}_hL_0 \frac{1}{h} (F_h\delta_x - F)$$

This is an indeterminate form in the last term for $h = 0$. The independent variable is h, B^{-1} is the multiplier h^{-1}, and A is $(F_h\delta_x - F)$, whence

$$D \cdot F \overset{S}{=} DF - {}_hL_0 h^{-1} (F_h\delta_x - F)$$

$$\overset{S}{=} DF - {}_hL_0 (\partial_h \cdot h)^{-1} \, \partial_h \cdot (F_h\delta_x - F)$$

$$\overset{S}{=} DF - {}_hL_0 \; \partial_h \cdot (F_h\delta_x)$$

since

$$\partial_h \cdot h \overset{S}{=} I$$

and

$$\partial_h \cdot F \overset{S}{=} \Omega \quad (F \text{ independent of } h)$$

where ∂_h denotes $\frac{\partial}{\partial h}$. F is a functor with regard to h and the derivative of a functor is the null operator (see previous paper).

*Altering the notation of the previous paper we define ${}_aL_b \overset{S}{=} \lim_{a \to b}$.

In the expression for $D \cdot F$ we may drop the dot on the right hand side of the equation since

$$\partial_h \cdot [F\, {}_h\delta_x\, f(x)] \equiv \partial_h F\, {}_h\delta_x\, f(x) \equiv \partial_h \cdot (F\, {}_h\delta_x)f(x) + F\, {}_h\delta_x\, \partial_h \cdot f(x)$$

but

$$\partial_h \cdot f(x) \equiv 0$$

and

$$\partial_h \cdot (F\, {}_h\delta_x)f(x) \equiv \partial_h F\, {}_h\delta_x\, f(x)$$

As an operator equation the above becomes

$$\partial_h \cdot (F\, {}_h\delta_x) \overset{S}{=} \partial_h F\, {}_h\delta_x$$

Therefore the derivative of a continuous operator may be expressed by

6.2 $$D \cdot F \overset{S}{=} DF - {}_hL_0\, \partial_h F\, {}_h\delta_x$$

To give a specific example of 6.2 for a non-linear, continuous operator we shall investigate

$$F \overset{S}{=} x \lg_e$$

as follows:

$$D \cdot (x \lg_e) \overset{S}{=} D x \lg_e - {}_hL_0\, \partial_h\, x \lg_e\, {}_h\delta_x$$

Applying this operator equation to an arbitrary operand, $f(x)$,

$$\begin{aligned} D \cdot (x \lg_e) f(x) &\equiv D x \lg_e \cdot f(x) - {}_hL_0\, \partial_h\, x \lg_e \cdot f(x+h) \\ &\equiv D x \lg_e \cdot f(x) - x\, {}_hL_0\, \partial_h \lg_e \cdot f(x+h) \\ &\equiv D x \lg_e \cdot f(x) - x\, {}_hL_0 \frac{f'(x+h)}{f(x+h)} \\ &\equiv \lg_e \cdot f(x) + x \frac{f'(x)}{f(x)} - x \frac{f'(x)}{f(x)} \\ &\equiv \lg_e \cdot f(x) \end{aligned}$$

Rewriting this as an operator equation

6.3 $$D \cdot (x \lg_e) \overset{S}{=} \lg_e$$

For instance, applying 6.3 to the specific function e^x,

$$D \cdot (x \lg_e) e^x \equiv \lg_e \cdot e^x \equiv x$$

As a second application of the Extended l'Hospital's Rule let us take the operator $h^{-1}(A^h - I)$ which is indeterminate at $h = 0$. Then

$${}_hL_0 \frac{1}{h}(A^h - I) \overset{S}{=} {}_hL_0 \frac{1}{\partial_h \cdot h} \partial_h \cdot (A^h - I)$$

But

$$\partial_h \cdot h \overset{S}{=} I$$

and

$$\partial_h \cdot I \overset{S}{=} \Omega$$

whence

$${}_hL_0 \frac{1}{h}(A^h - I) \overset{S}{=} {}_hL_0 \; \partial_h \cdot A^h$$

We may drop the dot in the right hand term by the same procedure used in deriving 6.2. Thus

6.4 $${}_hL_0 \frac{1}{h}(A^h - I) \overset{S}{=} {}_hL_0 \partial_h A^h$$

This operator will be further discussed in the next section.

7. The Logarithm of an Operator:

It is easily verified that

$${}_hL_0 \frac{a^h - 1}{h} \equiv \lg_e a$$

($\lg_e a$ can be written, consistent with 2.1, as $\lg_e \cdot a$). We define the logarithm of an operator, A, by

7.1 $$\lg_e \cdot A \overset{S}{=} {}_hL_0 \frac{1}{h}(A^h - I)$$

By equation 6.4 this becomes

7.2 $$\lg_e \cdot A \overset{S}{=} {}_hL_0 \partial_h A^h$$

As an example of the use of 7.2 let us define the addition operator, A_a, by

7.3 $$A_a f(x) \equiv f(x) + a$$

then

$$\lg_e \cdot A_a \overset{S}{=} {}_hL_0 \partial_h A_a^h$$

or, applying this to the arbitrary operand, $f(x)$,

$$\lg_e \cdot A_a f(x) \equiv {}_hL_0 \partial_h A_a^h f(x) \equiv {}_hL_0 \partial_h [f(x) + ah] \equiv {}_hL_0 a$$

7.4 $$\lg_e \cdot A_a f(x) \equiv a$$

Defining the Unifier operator, U, by

7.5 $$Uf(x) \equiv 1$$

i.e. that operator which converts all operands to unity, then writing 7.4 as an operator equation,

7.6 $$\lg_e \cdot A_a = aU$$

As a second example of the application of 7.2 let us use the inner addition operator defined in the previous paper:

$$_a\delta_x f(x) \equiv f(x+a)$$

Then, by 7.2,

$$\lg_e \cdot {}_a\delta_x \overset{S}{=} {}_hL_0 \partial_h {}_a\delta_x^h$$

or

$$\lg_e \cdot {}_a\delta_x f(x) \equiv {}_hL_0 \partial_h {}_a\delta_x^h f(x) \equiv {}_hL_0 \partial_h f(x+ah)$$

$$\equiv {}_hL_0 \, af'(x+ah) \equiv a f'(x)$$

whence

$$\lg_e \cdot {}_a\delta_x f(x) \equiv aD f(x)$$

or, as an operator equation,

7.7 $$\lg_e \cdot {}_a\delta_x \overset{S}{=} aD$$

In 7.4 and 7.7 we have the logarithms of two different types of addition operators and the results are similar yet interestingly different. In passing we note that the Unifier operator defined by 7.5 is related to the addition operator defined by 7.3 by the simple relation

7.8 $$U \overset{S}{=} A - I$$

Thus we can restate 7.6 as

7.9 $$\lg_e \cdot A \overset{S}{=} a(A-I)$$

It is possible to obtain an expression for the logarithm of a linear operator involving no limit procedure as follows: If B_l is a linear operator,

$$\partial_m \cdot B_l^m \stackrel{S}{=} {}_hL_0 \frac{1}{h} (B_l^{m+h} - B_l^m)$$

where B_l is independent of m. Then

$$\partial_m \cdot B_l^m \stackrel{S}{=} B_l^m \; {}_hL_0 \frac{1}{h} (B_l^h - I)$$

This step is only permissible if B_l is a continuous, linear operator. Furthermore

$$\partial_m \cdot B_l^m \stackrel{S}{=} B_l^m \lg_e \cdot B_l$$

by 7.1. It follows that

$$\lg_e \cdot B_l \stackrel{S}{=} B_l^{-m} \partial_m \cdot B_l^m$$

Again the dot in the right hand term may be dropped as before, giving

7.10 $$\lg_e \cdot B_l \stackrel{S}{=} B_l^{-m} \partial_m B_l^m$$

Sometimes equation 7.2 results in an indeterminate form at $h=0$, which may be eliminated by using 7.10. More will be said of this later.

8. The Logarithm of the Derivative:

By 7.2 the logarithm of the derivative is

8.1 $$\lg_e \cdot D \stackrel{S}{=} {}_hL_0 \partial_h D^h$$

Let us apply this to specific functions, e.g.

$$\lg_e \cdot D e^{ax} \equiv {}_hL_0 \partial_h D^h e^{ax} \equiv {}_hL_0 \partial_h a^h e^{ax} \equiv {}_hL_0 a^h \lg_e \cdot a e^{ax}$$

Thus

8.2 $$\lg_e \cdot D e^{ax} \equiv e^{ax} \lg_e \cdot a$$

Let us now apply $\lg_e \cdot D$ to $\sin x$. Since

$$\sin x = -\frac{i}{2} (e^{ix} - e^{-ix})$$

$$\lg_e \cdot D \sin x \equiv {}_hL_0 \partial_h D^h \sin x \equiv {}_hL_0 \partial_h D^h [-\frac{i}{2} (e^{ix} - e^{-ix})]$$

$$\equiv {}_hL_0 \partial_h \{-\frac{i}{2} [i^h e^{ix} - (-i)^h e^{-ix}]\}$$

$$\equiv {}_hL_0 \{-\frac{i}{2} [i^h \lg_e(i) e^{ix} - (-i)^h \lg_e(-i) e^{-ix}]\}$$

$$\equiv -\frac{i}{2}[\, e^{ix}\lg_e (i) - e^{-ix}\lg_e (\frac{1}{i})\,]$$

But

$$e^{n\pi i} = -1$$

and

$$\pm\, e^{n\pi i/2} = i$$

For the positive root,

$$\lg_e \cdot i = \frac{n\pi i}{2}$$

For the negative root

$$\lg_e \cdot i = \lg_e (-1) + \lg_e (e)^{n\pi i/2} = n\pi i + \frac{n\pi i}{2} = \frac{3}{2}\, n\pi i$$

which is a special case of the form obtained from the positive root. Thus

$$\lg_e \cdot D \sin x \equiv \frac{ni}{2}\,[\,\frac{i\pi}{2}\, e^{ix} + \frac{i\pi}{2}\, e^{-ix}\,]$$

or

8.3 $$\lg_e \cdot D \sin x \equiv \frac{n\pi}{2} \cos x$$

Similarly

8.4 $$\lg_e \cdot D \cos x \equiv -\frac{n\pi}{2} \sin x$$

Let us now apply $\lg_e \cdot D$ to x^n: Using 7.2,

$$\lg_e \cdot D\, x^n \equiv {}_hL_0\, \partial_h\, D^h x^n \equiv {}_hL_0\, \partial_h\, \frac{\Gamma(n+1)}{\Gamma(n-h+1)}\, x^{n-h}$$

$$\equiv {}_hL_0\, \Gamma(n+1) \left\{\frac{\partial_h\, \Gamma(n-h+1)}{\Gamma(n-h+1)^2}\right\} x^{n-h} - {}_hL_0\, \frac{\Gamma(n+1)}{\Gamma(n-h+1)}\, x^{n-h}\, \lg_e \cdot x$$

$$\equiv {}_hL_0 \left\{\frac{\Gamma(n+1)}{\Gamma(n-h+1)}\, \Psi(n-h+1) - \frac{\Gamma(n+1)}{\Gamma(n-h+1)}\, \lg_e \cdot x\right\} x^{n-h}$$

where

$$\Psi(n+1) \equiv \frac{\int_0^\infty e^{-x} x^n \lg_e \cdot x\, dx}{\Gamma(n+1)} \equiv \partial_n \lg_e \cdot \Gamma(n+1)$$

and

$$\Psi(1) \equiv -C \quad \text{(Euler's constant)}$$

and

$$\Psi(n+1) \equiv \Psi(n) + \frac{1}{n}$$

Then

8.5 $$\lg_e \cdot D\, x^n \equiv [\,\Psi(n+1) - \lg_e \cdot x\,]\, x^n$$

This result is different from the integral equation developments of $\lg_e \cdot D$. For example, Volterra*, by differentiation of p^{-n} with respect to n (p is the differential operator) obtains $-p^{-n}\lg_e \cdot p$ and takes the limit as $n \to 0$ to obtain $-\lg_e \cdot p$. Although this seems essentially the process employed by us, the result obtained is different, almost seeming to be a special case of 8.5. The reason for this lies in the basic concepts concerning operations. It is easy to show that the concept of operator equations as defined in the first paper cannot be used consistently in the theory of Integral equations. For instance, the logarithm of the derivative as defined by Volterra (symbolized by $\lg_{e_V} \cdot D$) has the following properties:

$$\lg_{e_V} \cdot D\, f(x) \equiv (-C - \lg_e \cdot x)\, f(x)$$

i.e.

$$\lg_{e_V} \cdot D\, x^n \equiv (-C - \lg_e \cdot x)\, x^n$$

This second equation is similar to equation 8.5 which, if n were an integer, could be written

$$\lg_e \cdot D\, x^n \equiv (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - C - \lg_e \cdot x)x^n$$

The difference between the two formulae is that the latter contains a harmonic series and the former does not. Because of the complexity of 8.5 it is not so readily extended to the application of $\lg_e \cdot D$ to $f(x)$ in general, although such an extension would be extremely useful.

It is shown in Davis "Theory of Linear Operators" that the Volterra $\lg_e \cdot D$ commutes with the differential operator, D, i.e.

$$D \lg_{e_V} \cdot D\, f(x) \equiv \lg_e \cdot D D\, f(x)$$

From our approach this is also indicated but if

$$\lg_{e_V} \cdot D\, f(x) \equiv (-C - \lg_e \cdot x)\, f(x)$$

then applying $\lg_{e_V} \cdot D$ is simply the operation of multiplying by $(-C - \lg_e \cdot x)$, i.e.

8.6 $$\lg_{e_V} \cdot D \overset{S}{=} (-C - \lg_e \cdot x)$$

and the application of this to e^{ax}, $\sin x$, and $\cos x$ would give results completely different from ours. Also the equation

*Davis: "The Theory of Linear Operators", Pgs. 78-80.

$$D\,\lg_{e_V} \cdot D \overset{S}{=} \lg_{e_V} \cdot DD$$

implies that

$$D(-C - \lg_e \cdot x) \overset{S}{=} (-C - \lg_e \cdot x)\,D$$

which is obviously impossible. This leads us to the conclusion that the basic definitions used in developing $\lg_e \cdot D$ by the Volterra integral equation approach are not consistent with the definitions used in operator equations, i.e. 8.6 cannot be written as such. This would seem to be a limitation on integral equations being considered as operator forms.

Three other interesting formulae that may be derived by the above methods are

8.7 $$\lg_e \cdot D\left(\frac{1}{x}\right)^n \equiv \left[\Psi(n) - \lg_e\left(-\frac{1}{x}\right)\right]\left(\frac{1}{x}\right)^n$$

8.8 $$\lg_e \cdot D\left(\frac{1}{1-x}\right)^n \equiv \left[\Psi(n) - \lg_e\left(\frac{1}{1-x}\right)\right]\left(\frac{1}{1-x}\right)^n$$

8.9 $$\lg_e \cdot D\,\lg_e \cdot x \mp \left[-C - \lg_e(-x^{-1/2})\right] \lg_e \cdot x$$

We must use 7.10 to derive 8.9 since using 7.2 results in an indeterminate form.

SEMI-POPULAR AND POPULAR PAGES

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The cosmopolitan nature of this group is indicated by the varied professions of a few of its members; a composer of piano music, a doctor, an automobile dealer, a member of the U.S. Embassy to Russia, member of the Metropolitan Grand Opera, men employed in industrial research laboratories, and of course teachers and students.

Already much has been accomplished by this group while it has been growing - a few original papers, some nice expository acticles and numerous problems and solutions of problems. Most important has been the increasing interest in mathematics on the part of laymen readers. This has instigated the establishment of a new department* called **Semi-popular and Popular Pages.**

This stage in the broadening** of the coverage of the Mathematics Magazine seems to be the proper time to make this statement, especially to those people who work (or play) with mathematics primarily because of their interest in it; such as the young man who recently sent us the following note."Next month I must return to school and I'll have no time to devote to any problems other than the assignments that the sterm professors punish us with"

EDITOR

* See Vol.28, No.1, pp,39-43.

** Extensions in the direction of clear, simple, correct discussions of most any mathematical concept are hardest to get. Experts are rare in this sort of work.

SCIENCE IN THE MODERN WORLD*

Marston Morse

There is no conflict between science, philosophy and theology. What conflict there may be is due to a failure of agreement as to the implications of the word 'science'.

I might illustrate my point by a short anecdote. In the 12th century, before learning was centralized in the great Universities of Paris, Oxford and Cologne, teachers of science were intellectual gypsies. They wandered about from place to place, stopping wherever they could gather an audience to dispense their knowledge. One of these intellectual gypsies was Roscelin, whom Peter Abelard worsted in intellectual joust. The story, often revamped, tells of a prank which was played upon Roscelin. Some local wags told him tall tales about the mental prowess of the local butcher. This butcher had but one eye, which was fact. He was dumb, literally dumb, which was fancy. But the jester told Roscelin that this butcher could converse in pantomine, by signs only, without words, and that in so doing he had worsted, had defeated the greatest of the scholars. Roscelin must have been a gullible sort, because he agreed to meet this sage of the crossroads.

Meanwhile the butcher had been informed, or misinformed, that Roscelin wished to examine him in sign language to discover if he, the butcher, were really the most ignorant of the louts. The bout was held. Not a word was spoken. Roscelin held up one finger; the butcher answered with two. Roscelin raised three fingers; the butcher a clenched fist. Roscelin showed him an orange; the butcher waved a crust of bread.

Recounting the encounter Roscelin said, 'I have never met such wisdom on all my travels. I held up one finger and told the butcher that God was one. He held up two fingers to remind me that God was Father and Son. With three fingers I proclaimed my faith in the Blessed Trinity, and he clenched his fist to assert the Unity of the Godhead. Finally I showed him an orange as proof that the world is round, and he waved before me a crust of bread to show me that bread is the staff of life.'

Meanwhile the butcher regaled his cronies. 'I cooked his goose. He sneered at my missing eye. I told him I could see more with my one eye than he with his two eyes. Then he said to me that we had only three eyes among us. I threatened then to punch his face. But the final blow was when he brought out a fruit from southern France to insult our village, and I told him what he could do with his fruit so long as we have good, wheaten bread.'

*From a talk delivered at a Symposium on the dedication of Albertus Magnus Hall of Science published here by courtesy of the College of St. Thomas, St. Paul, Minn.

I have sacrificed these few minutes because I believe that any quarrel in history, or today, between scientists, philosophers and theologians, is a quarrel of those who do not understand the signs, the meanings of the words they bandy. "--The Very Rev. John L. Callahan, O.P., Ph.D. (Phil.) Dominican House of Studies, River Forest, Ill."

The young men who study in Albertus Magnus Hall are likely to take an interest in a science proportional to their ability in that science, and their estimate of its importance. If that interest is to remain stable, it must be built on values which are permanently founded. It is sad to see men of talent in science, superficially convinced of the importance of a topic in science, abandon this topic for another within a few years. Whatever adjustments may be necessary to accord with the advances in science from day to day, it is abundantly clear that the traits of patience, persistence, and abiding integrity of scientific purpose are characteristic of the greatest scientists.

This continuity of effort is particularly important in mathematics. It is needed to realize the promise of unity which modern mathematics holds. In no science does it appear truer than in mathematics that the relatively unexplained universe of known facts can be unified by theories of a general character, built of the bricks of current techniques, if only there could rise enough men of talent with a sense of values that would hold them to their task to the very end.

The problem of values in science would have been urgent today even if the atomic age had not been born. The discussion of the values of science began with the Pythagoreans and has continued ever since, sometimes intense, sometimes diverse. Albert the Great and St. Thomas brought their knowledge to bear on the problem. Without the advent of the atomic age the mathematician of today would still be faced with a first formidable problem of choosing a field of research. From infinitely many possible paths of mathematical development it is not easy to select the most important, unless one regards all fields of equal importance. This infinite multiplicity of choice has always existed, even in antiquity, but the problem of choice has today become critical only because modern mathematics has opened infinitely many doors. The freedom of the will to choose evil or good is paralleled by the freedom of the intellect to choose the trivial or the significant, provided one but sees the difference. The vastness of the field of choice will be evident if I tell you that more has been accomplished in mathematics since 1900 than was accomplished in all of the years preceding 1900; this is not to disparage the work of the mathematicians who belong to the past.

With mathematics based on logic, and logic based on metamathematics, the diversity of math. appears at the very start. On these multiple foundations are built multiple fields of math., the new algebras, group theory and topology. Probability which once seemed obvious to everyone, but which was really clear to no one, is now for the first time placed on a logical instead of an intuitive basis. Born again, probability is confronted by its ancient relatives in the immense family of statistics. For the sake of its own integrity, probability must distinguish among its relatives, preferring the respectable and admonishing the disreputable. The electronic computing machine is here, with limited powers and unlimited publicity, serving some ends of science, and obscuring others.

The demands on mathematics of physics, chemistry, biology, and engineering are unlimited and can be met, at most, in part. There is the direct clinical approach, offering first aid with the mathematical instruments of today, and there is the deeper long term response, based on the conviction that mathematics will serve best if it first serves its own ideals of perfection. So mathematics approaches a horizon which forever recedes.

Small wonder, then, that a large proportion of the young mathematicians become technicians in limited fields mostly connected with the foundations. Some leap over the foundations and proceed at once to the front as represented by the material world; these are the ones whom we call applied mathematicians. They have good courage, but are frequently inadequately equipped, although with genius they may compensate for this deficiency. Then there are the few--all too few--who aim to build the whole edifice of mathematics, neither lingering too long over the foundations, nor too hastily testing their strength at the front. Such a mathematician was Riemann, who, fifty years before Einstein, built the structure in mathematics whose counterpart in physics is relativity. There is Hilbert, whose operations later made possible a first mathematical exposition of quantum mechanics. And there is Poincare, whose topological analysis has not yet come to its full fruition, but which I believe contains the germs of an unexpected and startling unity for analysis and parts of physics. I might add that this possibility is not known to the physicist, as far as I know, nor to the chemist. It is known only to a few mathematicians, and that is as it should be at the present.

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SOME ELEMENTARY PROPERTIES OF THE RELATION, CONGRUENCE, MODULO M*

Louis E. Diamond

Cyclic Permutation of the Integers, Modulo M

Given a sequence of n numbers, $A_1, A_2, \cdots, A_n$, the ordering $A_2, A_3, \cdots, A_n, A_1$ is said to be obtained from the original sequence by a cyclic permutation, that is, the order in which the numbers follow each other when placed around a circle and read in a certain direction. It is understood that, once chosen, this direction, clockwise or counter-clockwise, remains fixed. A cyclic permutation may also be considered as a substitution or replacement of each number by the one that follows it, the last number being replaced by the first. Thus, the order of the numbers around the circle is not changed and the numbers may be thought of as moving in the chosen direction. The three cyclic permutations for the letters A,B, and C are illustrated below.

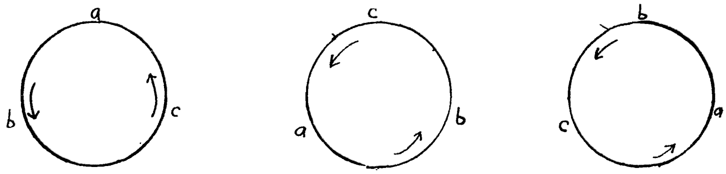


Figure 1.

Simply because a circle is used to illustrate the term, cyclic permutation should not be confused with the term "circular permutation". If three people seat themselves at a circular table, we consider "A" fixed, and then A can have either B or C at his right. The letters B and C are interchanged so that the cyclic order is not maintained.

In a cyclic permutation of three letters the original order is restored in three steps, or in general after n steps for a sequence of n numbers. This constitutes a "periodic" return to the original order.

If we take any m consecutive integers and divide each one by an integer m, then the remainders, taken in order, will be a cyclic permutation of the integers, $0, 1, 2, \cdots, (m-1)$. The integers thus exhibit a *periodicity* with respect to a given modulus. Consider $M=4$ and the following three random sets of four consecutive integers.

*Readers who are not pretty conversant with the concept, congruence, would profit by reading first Dr. Dimmick's article, Vol.28, No. 1, pg.41, then Prof. Richard V. Andree's article, Vol.28, No.3, pg.173 before reading this article. Editor.

4, 5, 6, 7	6, 7, 8, 9	65, 66, 67, 68
0, 1, 2, 3,	2, 3, 0, 1	1, 2, 3, 0

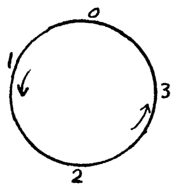


Figure 2.

In each case we divide by four and list the remainders in order. In each case this order is a cyclic permutation of the integers, 0,1,2,3.

The Relation, Congruence, is an Equivalence Relation

By an ordered pair of numbers is meant two paired numbers, (A,B) in which both the magnitude and the order of the two numbers are important. (A,B) is considered identical with (B,A) if, and only if, A is identical with B. The location of a point in the XY plane by an ordered number pair, $P(X,Y)$ is a common example. Consider a set of integers which are congruent, modulo M, in ordered number pairs. The ordered number pair, (A,B) is congruent in this order, $A \equiv B$, mod M. The logical or mathematical structure of this set depends both upon the elements and the manner in which these elements or terms are related to each other. A simple non-mathematical example consists of the two words, God and dog. They contain exactly the same letters or elements of word structure. The relations "before" and "after" of these letters are different in each case. The resulting words, of course, have completely different meanings.

For brevity, Modulo M will be considered to apply in this discussion without further mention. The logical properties of the congruence relation are the arrangements of elements that may or may not be made by means of it. Three important questions might be asked in this connection.

First, is $A \equiv A$? If so, the congruence relation is said to have a *reflexive property*. If $A \equiv A$, $A - A$, or zero, is divisible by M. Zero is divisible by any nonzero modulus, and hence $A \equiv A$.

Do not confuse this statement with the theorem that "division of any number by zero is impossible". (For real numbers $0 = B0$ for all numbers B. Hence, by definition 0 is divisible by B, an arbitrary nonzero number, with the unique result, zero. Of course, B can be zero but in that case we merely have the identity, $0 = 0$.) To divide by zero is equivalent to solving for X, the equation $0X = A$. For all numbers the left hand side is zero. Hence, the equation has no meaning except for $A = 0$. In this case the equation becomes $0X = 0$. Any number will suffice for X. Hence, the solution exists but it is not unique.

The second important question refers to the porperty known as *symmetry*. If $A \equiv B$, is $B \equiv A$? $A - B$ is divisible by M, hence $A - B = KM$ and $(B - A) = (-K)m$. Thus, the congruence relation possesses the property of symmetry. *Any relation, R, which combines terms, A and B, regardless of their order, possesses the property of symmetry,* i.e., if ARB always implies BRA, R possesses symmetry.

Finally this question is asked: If $A \equiv B$, and $B \equiv C$, is $A \equiv C$? This property is called *transitivity*. It is the property of being transferable from one number pair to another. It is very important since it may set up a chain of related terms. Thus, If $A \equiv B$, and it can be established that $B \equiv C$, then $A \equiv C$. If $C \equiv D$, then $A \equiv D$, etc. The proof follows.

1)		$A \equiv B(\text{mod}.M)$	or	$A = KM + B$	By definition
2)		$B \equiv C(\text{mod}.M)$	or	$B = SM + C$	By definition
3)	Then	$A = KM + SM + C$		(Substituting the value of B in 1).	
4)	Or	$A - C = M(K + S)$			
5)	$\therefore$	$A \equiv C$			

Any relation which has these three properties, *reflexivity, symmetry,* and *transitivity,* is called an *equivalence relation*. Hence, it is established that the relation, congruence, is an equivalence relation. Mathematical relations do not necessarily have these three properties. Three common mathematical relations will exemplify this in the following paragraphs.

Non-Equivalence Relations

Consider the non-zero set of integers. Let R_1 be the relation "is the negative off", and let A, B, and C be elements of the set. Since AR_1A is always false, R_1 is irreflexive in the set. This term, irreflexive, is a stronger term than non-reflexive. (If zero were not excluded from the set, the relation R_1 would be non-reflexive since for $A = 0$, AR_1A is not false.) The relation, R_1, is symmetric. Whenever AR_1B holds in the set, it implies that BR_1A holds. Observe the word "whenever". The fact that a relation is symmetric does not mean that for every number pair in the set, ARB is true. If it did mean this, we could write ARA for the number pair (A,A). Then the property of reflexivity would not be a property independent of symmetry. R_1 is transitive since for every A,B,C, in the set, whenever the conjunction AR_1B, BR_1C holds, then AR_1C holds.

Let R_2 be the relation "is divisible by" and let A,B, and C be elements of the non-zero set of integers. Since AR_2A is valid for every A in the set, R_2 is reflexive. The relation R_2 is non-symmetric since whenever AR_2B holds, it is not necessarily true that BR_2A. This is always the case when $A \neq B$. The relation, R_2, is transitive, since for every A,B,C, in the set the truth of the conjunction, AR_2B, BR_2C, implies AR_2C.

Let R_3 be the relation "is the square root of", and let A,B,C, be elements of the set of positive integers greater than 1. R_3 is irreflexive in the set since AR_3A is false for every element A in the set. R_3 is asymmetric in the set since for every A,B in the set it is false that *both* AR_3B and BR_3A. The relation is intransitive. A true proposition can be set up, AR_3B, BR_3C as, for example, $2 = \sqrt{4}$, $4 = \sqrt{16}$, but this true conjunction never implies AR_3C, or in the example given, that $2 = \sqrt{16}$. The relation R_3 between A and C never holds when AR_3B and BR_3C are valid.

Modular Addition and Multiplication

The method by which elements of a set are combined or associated to form other elements of the set is called an operation. A binary operation involves the association of two elements to form a third element. for example, elementary arithmetic deals almost exclusively with positive rational numbers and with two basic operations, addition and multiplication. For each given pair of numbers these two operations each determine a unique number called respectively their sum and their product. In many branches of mathematics it is customary, by analogy, to call certain binary operations addition and multiplication even though they are not at all like ordinary arithmetical addition and multiplication. Similarly the symbols + and x are used for these operations even though they do not have their ordinary arithmetical meaning. A striking example is vector addition.

In mathematics definitions are logically arbitrary. In other words, they are neither proven nor disproven but they must admit of no contradictions. The definition of modular addition, for example, must combine two residues in such a manner that there is a unique result or sum for a given modulus. For a given modulus the sum of two residues is the residue to which the arithmetic sum of the numbers is congruent. The arithmetic sum of 10 and 12 is 22. 22 is congruent to 9, modulus 13. Hence in modular addition the sum of 10 and 12 is 9, modulo 13. Due to the periodicity of the integers discussed in a preceding paragraph, if X is any integer whatsoever, congruent to 10, and Y is any integer whatsoever, congruent to 12, and Z is any integer whatsoever, congruent to 9, then the modular sum of X and Y is congruent to Z, modulo 13.

The product of two residues is the residue to which the arithmetic product is congruent. The arithmetic product of 10 and 12 is 120. This product is congruent to 3, modulo 13. Hence the modular product of 10 and 12 is 3, modulo 13. If X and Y are defined exactly as in the preceding paragraph, and Z is defined as any integer whatsoever, congruent to 3, then the modular product of X and Y is congruent to Z, modulo 13.

From the definitions of addition and multiplication, the following theorems now follow.

1. If $A \equiv B$, $A + X \equiv B + X$ for all integers X
2. If $A \equiv B$, $AX \equiv BX$ for all integers X
3. If $A \equiv B$ and $C \equiv D$, then $A + C \equiv B + D$
4. If $A \equiv B$ and $C \equiv D$, then $A - C \equiv B - D$
5. If $A \equiv B$ and $C \equiv D$, then $AC \equiv BD$

Proof of 5

$AC \equiv BC$ from 2

$BC \equiv BD$ From 2

$AC \equiv BD$ from transitivity property

Theorems

The largest integer, D, which divides both the integers C and M is called their greatest common divisor. This is symbolized by $(C,M) = D$. If $D = 1$, C and M are said to be relatively prime, i.e., they have no common divisor except unity.

THEOREM: If $CA \equiv CB \pmod{M}$, $(C,M) = D$, $M = DW$, then $A \equiv B \pmod{W}$

Proof: By hypothesis C is divisible by D. Set $C = DL$

By hypothesis $(C,M) = D$, and $M = DW$. Substituting $(DL,DW) = D$, hence $(L,W) = 1$. $CA = DLA$ $CB = DLB$ $(CA - CB) = DL(A - B)$

By definition $(CA - CB)$ and hence $DL(A - B)$ is divisible by $M = DW$, or $L(A - B)$ is divisible by W.

Since $(L,W) = 1$, $(A - B)$ is divisible by W and by definition $A \equiv B \pmod{W}$.

Example: $104 \equiv 20 \pmod{12}$
$4 \cdot 26 \equiv 4 \cdot 5 \pmod{12}$
$26 \equiv 5 \pmod{3}$

THEOREM: $A \equiv B \pmod{M}$, if, and only if, A and B leave the same remainder R when divided by M. $0 \leq R < M$

The words "if and only if" show that there must be two parts to this proof.

1. The fact that A is congruent to B, mod M, implies that A and B leave the same remainder, R, when divided by M, i.e., a necessary condition that A is congruent to B is that A and B leave the same remainder R when divided by M.

Proof: Let $A = MQ + R$ when divided by M.

$A - B = KM$ by definition

$B = A - KM$ Transposition

$B = MQ + R - KM$ Substitution for A

$B = M(Q - K) + R$, when divided by M.

2. The fact that A and B leave the same remainder, R, when divided by M, implies that A is congruent to B, mod M, i.e., a sufficient condition that A is congruent to B is that A and B leave the same remainder R when divided by the modulus, M.

Proof: Let $A = MQ_1 + R$, $B = MQ_2 + R$.

$A - B = M(Q_1 - Q_2)$

Therefore $A \equiv B (\text{mod } M)$

Hence, $A \equiv B$ implies the remainder condition and the remainder proposition implies the congruence condition. The two conditions mutually imply each other and are logically equivalent.

$$121X \equiv 5 (\text{mod } 12).$$

$121X$ divided by 12 gives $(12)(10)X + X$, where X is the remainder.

5 divided by 12 is 0 with remainder 5.

$$\therefore X \equiv 5 (\text{mod } 12)$$

Addition and Multiplication Tables, Modulo 6

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

To find the sum of 5 and 3, locate in the addition table the row headed 5 at the left side. Go across this row to the right until the column is reached which is headed 3 at the top. There the number 2 is found and this is the modular sum of 5 and 3. The modular product of 5 and 3 is similarly found in the multiplication table.

The Closure Property of the Tables

Every integer, modulo 6, is congruent to one and only one of the integers, 0,1,2,3,4,5. All integers congruent to each other, modulo 6, are said to belong to the same residue class, modulo 6. The above tables thus represent in a sense all residue classes, modulo 6, by the symbols 0,1,· · ·,5. The sum or the product of any two integers can be reduced to one of the above six symbols. The tables show that the sum of any two of these symbols is invariably another one of these symbols. The product of any two of these symbols is invariably another one of these symbols. This fact is formally expressed by stating that the set of integers, 0,1,· · ·,5 is "closed" under the operations of addition and multiplication, modulo 6. Closure can be usefully regarded as representing *completion in itself,* a sort of self - sufficiency. This very simple concept of closure occupies a most important position in mathematics.

Properties of the Operations, Addition and Multiplication

For integers, and in fact for all real numbers, there are five important properties which refer to addition and multiplication. These properties also hold for modular addition and multiplication. Examples of these properties for integers and for the system, modulo 6, are given below.

An operation is associative if, when the operation is used on three or more elements, the result is independent of the grouping.

Arithmetic Addition	Modulo 6 Addition
$5 + (2 + 3) = (5 + 2) + 3$	$5 + (2 + 3) \equiv (5 + 2) + 3$
$5 + 5 = 7 + 3$	$5 + 5 \equiv 1 + 3$
$10 = 10$	$4 \equiv 4$

Arithmetic Multiplication	Modulo 6, Multiplication
$(5)(3)\ 2 = 5\ (3)(2)$	$(2)(4)\ 5 \equiv 2\ (4)(5)$
$(15)2 = 5(6)$	$(2)5 \equiv 2(2)$
$30 = 30$	$4 \equiv 4$

An operation is commutative if the operation used on two arbitrary elements taken in a certain order gives a result identical with that obtained when the operation is used on the same two elements taken in the opposite order.

Arithmetic Addition	Modulo 6 Addition
$5+2=2+5$	$5+2\equiv 2+5$
$7=7$	$1\equiv 1$
Arithmetic Multiplication	**Modulo 6, Multiplication**
$(5)(3)=(3)(5)$	$(5)(3)\equiv(3)(5)$
$15=15$	$3\equiv 3$

Multiplication is distributive with respect to addition. Symbolically $A(B+C)=AB+AC$. This property connects the two operations.

Arithmetic	Modulo 6
$5(3+2)=(5)(3)+(5)(2)$	$5(3+2)\equiv(5)(3)+(5)(2)$
$(5)(5)=15+10$	$(5)(5)\equiv 3+4$
$25=25$	$1\equiv 1$

Conclusions

By changing logically the meaning of certain symbols and operations, *different true conclusions* can be reached. The average person is quite apt to believe that the statements $2+2=4$, and $2+2=4$ are *unqualifiedly* true. About a year ago a certain firm headed their advertisement in a mathematical journal with "as sure as $2\times 2=4$". However, in modulo 4 addition and multiplication tables both the sum and the product of two 2's are the zero symbol. Of course, this merely states that the sum and the product are multiples of 4 with remainder zero.

A DIVISION ALGORITHM WITH NUMBER PAIRS

N. A. Draim

Algorithm for developing f in the identity $(P + f)(Q + f) = N$ in terms of an infinite, periodic, simple continued fraction where P, Q, N are positive integers. Let $N = PQ + D_1$. Select P and Q such that D_1, the remainder, is equal to, or less than, $P + Q_1$ in order that f may be less than 1. (If $f = 1$, then $N = (P + 1)(Q + 1) = PQ + (P + Q) + 1$) P may equal Q in the special case but for convenience and generality take $Q \geq P$ and call $Q-P$, δ.

The method will be illustrated by a numerical example.

Example: Find f, in the expression $(5 + f)(13 + f) = 79$.

$$79 = 5 \times 13 + 14.$$

$$P = 5;\ Q = 13;\ D_1 = 14;\ \text{and}\ \delta = 13 - 5 = 8.$$

$$D_1 < P + Q.$$

STEP 1: Find the first quotient, Q_1, by dividing the sum of the components of the first number pair (P_1Q_1) by D_1 and take as Q_1 the nearest integral quotient with positive remainder. Indicate this operation by brackets, [ ].

Thus, $$Q_1 = \left[\frac{P+Q}{D_1}\right] = \left[\frac{5+13}{14}\right] = \left[\frac{18}{14}\right] = 1.$$

Compute R_1, $= D_1Q_1 - Q$.

$$R_1 = 14 \times 1 - 13 = 1.$$

STEP 2: Find the second divisor, D_2, such that

$$D_2 = \frac{N - R_1\ (R_1 + \delta)}{D_1}.$$

$$D_2 = \frac{79 - 1\ (1+8)}{14} = \frac{70}{14} = 5.$$

STEP 3: Find the second quotient, Q_2, such that

$$Q_2 = \left[\frac{P + R_1 + \delta}{D_2}\right]$$

$$Q_2 = \left[\frac{5 + 1 + 8}{5}\right] = \left[\frac{14}{5}\right] = 2.$$

Compute $R_2 = D_2Q_2 - R_1 = 5 \times 2 - 1 = 9$.

STEP 4: Find the third divisor, D_3, such that

$$D_3 = \frac{N - R_2\ (R_2 - \delta)}{D_2} .$$

$$D_3 = \frac{79 - 9\ (9-8)}{5} = \frac{70}{5} = 14 .$$

STEP 5: Find the third quotient, Q_3, such that

$$Q_3 = \left[\frac{P + R_2}{P_3}\right] = \left[\frac{5 + 9}{14}\right] = 1 .$$

Compute $R_3 = D_3Q_3 - R_2 = 14 \times 1 - 9 = 5$.

* * * * * *

The number recurrences and sequences are now established and the successive divisors, quotients, and remainders may be derived in a step by step process which turns out to be periodic and non-ending. Note that

$$Q_n = \left[\frac{P + R_{n-1}}{D_n}\right] \quad \text{when } n \text{ is odd,}$$

and

$$Q_n = \left[\frac{P + R_{n-1} + \delta}{D_n}\right] \quad \text{when } n \text{ is even.}$$

$$D_n = \frac{N - R_{n-1}\ (R_{n-1} - \delta)}{D_{n-1}}, \quad \text{when } n \text{ is odd,}$$

and

$$D_n = \frac{N - R_{n-1}\ (R_{n-1} + \delta)}{D_{n-1}}, \quad \text{when } n \text{ is even.}$$

* * * * * *

The arithmetical development of the example algorithm in the condensed form in which it would be calculated appears as follows:

```
            5 ) 79  | 13
                65
D1 =       14 ) 5, 13 | 1 = Q1
                   14
D2 =        5 ) 5,  1 | 2 = Q2
 :                 10       :
           14 ) 5,  9 | 1
                   14
            1 ) 5,  5 | 18
                   18
D5 =       14 ) 5, 13 | 1 = Q5.
                        :
```

Note that $D_5 = D_1 = 14$, and $Q_5 = Q_1 = 1$, and the fifth number pair 5, 13 = the first. The computations therefore, if continued, would be infinitely periodic.

Take the successive quotients, $Q_1, Q_2, Q_3 \ldots$ computed by the foregoing algorithm, as the successive denominators of a simple, periodic, continued fraction. This fraction is equal to f which was to be found.

$$f = \left[\cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{18 + \cfrac{1}{1 \cdots}}}}}\right] = \left[0,\ \mathring{1},\ 2,\ 1,\ 1\mathring{8}\right]$$

and $79 = (5,\ \mathring{1},\ 2,\ 1,\ 1\mathring{8})(13,\ \mathring{1},\ 2,\ 1,\ 1\mathring{8})$.

* * * * * *

The development of the fractional part of the square root of an integer not a perfect square is a special case of the foregoing algorithm in which $P = Q$ and $Q - P = \delta = 0$. It is thought desirable to illustrate this special case in view of its importance.

Example: Develop f, in the expression $(8 + f)^2 = 79$. The algorithm, taking $N = 79$; $P = Q = 8$; $\delta = 0$; and $D_1 = 15$, appears as follows:

```
 8 ) 79 | 8
     64
    15 ) 8, 8 | 1
           15
     2 ) 8, 7 | 7
           14
    15 ) 8, 7 | 1
           15
     1 ) 8, 8 | 16
           16
    15 ) 8, 8 | 1
              :
```

$$Q_1 = \left[\frac{8+8}{15}\right] = \left[\frac{16}{15}\right] = 1$$

$$R_1 = 15 - 8 = 7$$

$$D_2 = \frac{79 - 7^2}{15} = 2$$

$$Q_2 = \left[\frac{8+7}{2}\right] = \left[\frac{15}{2}\right] = 7$$

$$R_2 = 14 - 7 = 7$$

$$D_3 = \frac{79 - 7^2}{2} = 15, \qquad \text{etc.}$$

Then,

$$\sqrt{79} = \left[8 + \cfrac{1}{1 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{16 + \cfrac{1}{1 \cdots}}}}}\right] = [8,\ \mathring{1},\ 7,\ 1,\ 1\mathring{6}]$$

* * * * * *

The foregoing algorithm can be extended to the algebraic field. It appears that every integer is a term in one or more quadratic form, in x (with $x = 0, 1, 2, \ldots$), which can be treated by the algorithm. Only some quadratics of a certain form are amenable to treatment. For example: by using the algorithm, the following identities can be developed:

$$(24x^2+46x+16) = (4x+2,\dot{1},2x,1,\dot{10x+8})(6x+6,\dot{1},2x,1,\dot{10x+8})$$

and

$$\sqrt{4x^2+12x+5} = (2x+2,\ \dot{1},\ x,\ 2,\ x,\ 1,\ \dot{4x+4})$$

$$x = 0,\ 1,\ 2,\ \ldots .$$

* * * * * *

The Generalized Algorithm:

Consider $N = PQ + D_1 = (P + f)(Q + f)$. P and Q are chosen so that $0 < f < 1$. This is the case when $D_1 \gtreqless P + Q$. Take $Q > P$ and set $Q - P = \delta$.

Development:

```
P ) N  | Q
    PQ
    D1 ) P, Q     | Q1
          D1Q1
    D2 ) P, R1    | Q2
          D2Q2
    D3 ) P, R2    | Q3
    :               :
    Dn ) P, Rn-1  | Qn
    :               :
```

* * * * * *

Number Recurrences, Relationships, and Identities: In the following identites, in order to indicate proper signs, use is made of $i = \sqrt{-1}$. Thus, $i^{2n} = +1$ where n is even and $i^{2n} = -1$ where n is odd.

I. $R_1 = D_1Q_1 - Q$

$R_2 = D_2Q_2 - R_1 = D_2Q_2 - D_1Q_1 + Q$

\- - - - -

$R_n = D_nQ_n - R_{n-1} = D_nQ_n - D_{n-1}Q_{n-1} + \cdots - i^{2n}.\ D_1Q_1 + i^{2n}Q$

II.

$$D_1 = N - PQ$$

$$D_2 = \frac{N - R_1(R_1 + \delta)}{D_1}$$

$$D_3 = \frac{N - R_2(R_2 - \delta)}{D_2}$$

- - - - - - -

$$D_n = \frac{N - R_{n-1}(R_{n-1} + i^{2n}\delta)}{D_{n-1}}$$

III.

$$Q_1 = \left[\frac{P + Q}{D_1}\right] = \frac{R_1 + Q}{D_1}$$

$$Q_2 = \left[\frac{P + R_1 + \delta}{D_2}\right] = \frac{R_2 + R_1}{D_2}$$

$$Q_n = \left[\frac{P + R_{n-1} + (1 + i^{2n})\frac{\delta}{2}}{D_n}\right] = \frac{R_n + R_{n-1}}{D_n}$$

IV.

$$D_1 = N - PQ$$

$$D_2 = 1 - D_1Q_1^2 + 2Q_1Q - Q_1\delta$$

$$D_3 = D_1 - D_2Q_2^2 + 2Q_2R_1 + Q_2\delta$$

- - - - - -

$$D_n = D_{n-2} - D_{n-1}Q_{n-1}^2 + 2Q_{n-1}R_{n-2} - i^{2n}Q_{n-1}\delta$$

V.

$$D_1 = N - PQ$$

$$D_2 = 1 - D_1Q_1^2 + Q_1(P + Q) = 1 + D_1Q_1^2 - Q_1(2R_1 + \delta)$$

$$D_3 = D_1 - D_2Q_2^2 + 2Q_2D_1Q_1 - Q_2(P + Q)$$
$$= D_1 + D_2Q_2^2 - Q_2(2R_2 - \delta)$$

$$D_4 = D_2 - D_3Q_3^2 + 2Q_3D_2Q_2 - 2Q_3D_1Q_1 + Q_3(P + Q)$$
$$= D_2 + D_3Q_3^2 - Q_3(2R_3 + \delta)$$

- - - -

$$D_n = D_{n-2} - D_{n-1}Q_{n-1}^2 + 2Q_{n-1}D_{n-2}Q_{n-2} - \dots + i^{2n}Q_{n-1}(P + Q)$$
$$= D_{n-2} + D_{n-1}Q_{n-1}^2 - Q_{n-1}(2R_{n-1} + i^{2n}\delta).$$

VI. Let $Q\ Q_1\ Q_2 \dots Q_n$ and $P\ Q_1\ Q_2 \dots Q_n$ be the operators for generating the successive terms of two convergent simple continued fractions, (Hardy and Wright, "Theory of Numbers", 1938, Chap. X, Arts. 10.1, 10.2, 10.3), as follows:

$n =$	-2,	-1,	0,	1,	2, ...	n-2,	n-1,	n
			Q	Q_1	Q_2	Q_{n-2}	Q_{n-1}	Q_n
$\psi =$	0,	1,	$\underline{Q}$	$\underline{Q_1Q+1}$	$\underline{Q_2(Q_1Q+1)+Q}$ ..	$\underline{\psi_{n-2}}$	$\underline{\psi_{n-1}}$	$\underline{\psi_n}$

$$\begin{array}{lllllllll}
\theta = & 1, & 0, & 1 & Q_1 & Q_2Q_1+1 & \theta_{n-2} & \theta_{n-1} & \theta_n \\
 & & & P & Q_1 & Q_2 \; \cdots\cdots & Q_{n-2} & Q_{n-1} & Q_n \\
\phi = & 0, & 1 & P & Q_1P+1 & Q_2(Q_1P+1)+P \;\cdots & \phi_{n-2} & \phi_{n-1} & \phi_n \\
\hline
\theta = & 1, & 0 & 1 & Q_1 & Q_2Q_1+1 & \theta_{n-2} & \theta_{n-1} & \theta_n
\end{array}$$

In the above the following terms are written arbitrarily:

$$\psi_{-2} = 0; \; \psi_{-1} = 1; \; \phi_{-2} = 0; \; \phi_{-1} = 1; \; \theta_{-2} = 1; \; \theta_{-1} = 0.$$

Subsequent terms are calculated using the following relationships:

$$\psi_n = Q_n \, \psi_{n-1} + \psi_{n-2}$$
$$\phi_n = Q_n \, \phi_{n-1} + \phi_{n-2}$$
$$\theta_n = Q_n \, \theta_{n-1} + \theta_{n-2}$$

$$\frac{\psi_0}{\theta_0} = \frac{Q}{1} = Q$$
$$\frac{\psi_1}{\theta_1} = \frac{Q_1Q+1}{Q_1} = Q + \frac{1}{Q_1}$$
$$\frac{\psi_2}{\theta_2} = \frac{Q_2(Q_1Q+1)+Q}{Q_2Q_1+1} = Q + \frac{Q_2}{Q_2Q_1+1} \; \cdots\cdots$$

- - -

$$\frac{\psi_n}{\theta_n} = Q + F_n,$$

where F_n = a rational proper fraction the convergent of,

$$N\left[0, \; Q_1, \; Q_2 \; \cdots\cdots \; Q_n\right]$$

and

$$0 < F_n < 1 \; .$$

Similarly,

$$\frac{\phi_n}{\theta_n} = P + F_n.$$

The following relationships occur subject to inductive proof:

VII.

$$\psi_1 - \phi_1 = (Q-P)\theta_1 = \delta\theta_1$$
$$\psi_2 - \phi_2 = \delta\theta_2$$

- - -

$$\psi_n - \phi_n = \delta\theta_n \; .$$

VIII. $$D_1 = N - PQ.$$

$$D_2 = 1 - D_1Q_1^2 + Q_1(\psi_0\phi_{-1}+\psi_{-1}\phi_0) - 2Q_1\theta_0\theta_{-1}N$$

$$D_3 = D_1 - D_2Q_2^2 - Q_2(\psi_1\phi_0+\psi_0\phi_1) + 2Q_2\theta_1\theta_0N$$

- - - - -

$$D_n = D_{n-2}Q_{n-1}^2 + i^{2n}\left[(\psi_{n-2}\phi_{n-3}+\psi_{n-3}\phi_{n-2}) - 2Q_{n-1}\theta_{n-2}\theta_{n-3}N\right]$$

IX. $$(\psi_1\phi_0+\psi_0\phi_1) - 2\theta_1\theta_0N = -2D_1Q_1 + (P+Q) = -2R_1 - \delta.$$

$$(\psi_2\phi_1+\psi_1\phi_2) - 2\theta_2\theta_1N = 2D_2Q_2 - 2D_1Q_1 + (P+Q) = +2R_2 - \delta.$$

- - - -

$$(\psi_n\phi_{n-1}+\psi_{n-1}\phi_n) - 2\theta_n\theta_{n-1}N = i^{2n}(2D_nQ_n - 2D_{n-1}Q_{n-1}\cdots) + (P+Q)$$
$$= i^{2n}.2R_n - \delta.$$

X. $$D_1 = -PQ + N$$

$$D_2 = (PQ_1+1)(QQ_1+1) - NQ_1^2 = \phi_1\psi_1 - N\theta_1^2.$$

- - -

$$D_{n+1} = -i^{2n}(\phi_n\psi_n - N\theta_n^2).$$

* * * * * *

XI. From X above,

$$N = \frac{\phi_n\psi_n}{\theta_n^2} - i^{2n}.\frac{D_{n+1}}{\theta_n^2}\ \ldots$$

In the numerical calculation of the algorithm, we impose upon D_{n+1} an upper bound which we take as $P + Q$. But θ_n^2 becomes and remains larger than any integer, however large.

$$\therefore \lim_{n\to\infty} i^{2n}\frac{D_{n+1}}{\theta_n^2} = 0.$$

It follows that

$$\lim_{n\to\infty}\frac{\phi_n\psi_n}{\theta_n^2} = N.$$

From VI above,

$$\frac{\psi_n\phi_n}{\theta_n^2} = (Q + F_n)(P + F_n).$$

$$\therefore N = \lim_{n\to\infty}(Q + F_n)(P + F_n) = (Q + f)(P + f).$$

It follows that

$$\sqrt{N} = \lim_{n\to\infty} \frac{\sqrt{\phi_n \psi_n}}{\theta_n}.$$

$$(Q + f)(P + f) = N = f^2 + f\,(P + Q) + PQ.$$

f, then, is a quadratic surd.

By Theorem 177, Hardy and Wright, "Theory of Numbers", that "*The continued fraction which represents a quadratic surd is periodic*", f is necessarily periodic.

XII. Let $\delta = 2m$.
Then

$$(P + f)\ (Q + f) = (P + f)(P + 2m + f) = N.$$

$$\therefore f^2 + 2f\,(m + P) + (m + p)^2 = N + m^2\ ,$$

and

$$f = -\ (m + P) + \sqrt{N + m^2}\ .$$

It follows that the algorithms for

$$P\,\overline{)\ N\ }\!\underline{\lfloor\ Q\ }\ ,$$

and

$$m + P\,\overline{)\ N + m^2\ }\!\underline{\lfloor\ m + P\ }$$

have identical D_n's and Q_n's, respectively.
When

$$\delta = 0,\ P = Q, \text{ and } m = 0.$$

$$\phi_n = \psi_n; \text{ and } N = P^2 + D_1\ .$$

Then

$$f = -P + \sqrt{N}\ ,$$

and

$$\sqrt{N} = \lim_{n\to\infty} \frac{\phi_n}{\theta_n}\ .$$

Thus, $\sqrt{N}$ is a special case of XI, above.

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H V Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

The Parabola of Surety, A Sequel

J. H. Butchart

Recently R. F. Graesser showed that the parabola of surety (the boundary beyond which an airplane would be safe from an antiaircraft gun) can be established without recourse to calculus. He also noted that the gun is at the focus of this parabola. The purpose of this note is to derive this result and some other properties of the figure in another manner which may be new.

We first show that the distance $P\alpha$ from the focus to the directrix of a parabola traversed by a projectile shot with initial velocity v_0 and angle of elevation α is $(v_0^2 \cos^2\alpha)/g$. Note that the horizontal component of the velocity is constant being $v_0 \cos\alpha$ When the bullet passes upward through one end of the latus rectum, its vertical velocity component, excluding gravity, is the same, $v_0 \cos\alpha$. At the highest point, this upward velocity is balanced by the speed of fall acquired by the body during the same interval, so we equate $v_0 \cos\alpha$ and gt, and thus get $(v_0 \cos\alpha)/g$ as the length of this interval in seconds. Substituting this in the formula for the height, $y = (v_0 \cos\alpha)t - gt^2/2$, we can locate the vertex and then determine that $p_\alpha = v_0^2 \cos^2\alpha/g$.

Suppose now that we conjecture the envelope of these trajectories to be parabolic with the gun at its focus. The highest point reached by a shot fired vertically would be $v_0^2/(2g)$, derived as above by equating rising and falling velocities, so the parameter p for this envelope is v_0^2/g. Thus the similar snd similarly placed trajectory and envelope are homothetic with the homothetic ratio p_α/p equal to $\cos^2\alpha$. It is interesting to note that if the projectile is fired horizontally, the trajectory is congruent to the envelope, and if the angle of elevation is 45^0, it is half as large.

Take the equation of the conjectured envelope e, in polar coordinates to be $r = p/(1 - \cos\theta) = (p/2)\csc^2\theta/2$, where the pole is at the focus and the polar axis is positive downwards. At the point (r,θ) of e, construct a parabola tangent to this one, homothetic to it, and passing through the pole. The point of contact P is clearly the homothetic center. From known properties, the normal from the focus O to the tangent at P meets it at T on the tangent to e at the vertex V of e. Also OT extended meets the directrix at D, the projection of P on the directrix, OT equalling TD. Let PO extended meet e again at Q. It is known that the tangents at P and Q, the ends of a focal chord, are perpendicular and meet on the directrix. Thus by homothecy, OT is tangent to t, the trajectory, VT is its directrix , and the focus F of t is on OP. From an easy study of the figure, we conclude that the angle of elevation α of OT is half the angle OPD, or $\theta/2$, where θ is the angle for P in polar coordinates. Then OP is $(p/2)\csc^2\theta/2$, OQ is $(p/2)\sec^2\theta/2$, and PO/PQ or $PO/(PO+OQ)$ equals $\cos^2\theta/2$ or $\cos^2\alpha$. Referring to the known ratio $p_\alpha/p = \cos^2\alpha$, we see that the parabola t is indeed the trajectory.

The locus of the focus F of t as α changes is of interest. By homothecy, $OF/OP = QO/QP = \sin^2\alpha$. Thus $OF = (p/2)\csc^2\alpha\sin^2\alpha = p/2$. Hence F moves on a circle with O as center and tangent to e at V. Since the vertex of t is halfway from F to the directrix VT, this vertex moves on an ellipse with major axis p and minor axis $p/2$.

Arizona State College

Mathematics of Engineering Systems, By Derek F. Lawden, John Wiley & Sons, Inc., 440 Fourth Ave., New York 16, New York, 1955, 380 pp., $5.75.

The book gives an account of several mathematical methods used to analyze the behavior of various physical systems. Written by Derek F. Lawden, the new book was prepared for use in the fields of electronics and electrical engineering, applied physics, and instrument technology.

The author first deals with linear differential equations with constant coefficients I (Classical Methods) and those with constant coefficients II (Modern Methods). He then goes into Fourier analysis, and concludes with a discussion of non-linear differential equations.

Although the emphasis is on these mathematical methods, the author does not view them entirely from the abstract. Each is employed to solve a number of practical engineering problems, with other possible uses suggested by practically slanted exercises.

Richard Cook

Elementary Statistics, John M. Howell and Ben K. Gold, Wm. C. Brown Company, 1954, $3.00.

Elementary Statistics is designed as a text for a one semester course in beginning modern statistics. It is aimed at the non-mathematical student who needs a basic understanding of the principles of analyzing data in whatever field the data occurs. Explanations are kept as brief as possible and amplified with illustrations. Demonstrations and laboratory experiments are suggested to convince the student of the truth of basic principles which cannot be derived without extensive mathematical training. There are numerous problems drawn from many fields. A complete Glossary of symbols is included in the appendix together with tables and index. Chapter Titles are: Introduction; Numbers; Measurements-Quantitative Data; Attributes - Qualitative Data; The Normal Distribution; Estimation, Confidence Limits, and Significance Tests for Large Sample Sizes; The Chi-Square Distribution; Correlation; Descriptive Statistics.

John M. Howell and Ben K.Gold

Free Mathematics Paper

The *University of Oklahoma Mathematics Letter* is a four-page publication of interest to high school and beginning college students and teachers. Copies will be sent without charge to persons requesting them and enclosing a stamped self addressed envelope plus a 3″ x 5″ card giving their name and school address. Send all requests to Professor Richard V. Andree, Department of Mathematics, The University of Oklahoma, Norman, Oklahoma. You owe it to your students to send for this interesting free publication.

Summer Mathematics Institute

The *1955 Summer Institute for Mathematics Teachers* will be held in air-conditioned classrooms at the University of Oklahoma in Norman, Oklahoma from June 6 to June 17 . Teachers may attend either week (one hour credit) or both weeks (two hours credit). The fee is $15 for one week or $25 for two weeks - either credit or non-credit. Official certificates of attendance will be issued. Dormitory accommodations are available at $2 per night or $10 per week.

Leaders in the teaching of Elementary, Junior High School, High School, and beginning College Mathematics will be on hand to assist teachers with individual problems related to both classroom procedures and subject matter. Material for the interest and enrichment of standard courses will be provided.

If you or any of your colleagues are interested, please send a post card now to F. Lee Hayden, Short Courses and Conferences, the University of Oklahoma, Norman, Okla.

PROBLEMS AND QUESTIONS

***Edited** by*

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 N. Vermont Ave., Los Angeles, 29, California.*

PROPOSALS

229. *Proposed by Chih-yi Wang, University of Minnesota.*

Let AD be a median of a triangle ABC. If CE is perpendicular to AD and angle ACE equals angle ABC, prove, geometrically that either $AB = AC$ or angle BAC is a right angle.

230. *Proposed by John M. Howell, Los Angeles City College.*

Prove

$$\sum_{x=0}^{2n} (-1)^x \binom{2n}{x}^2 = (-1)^n \binom{2n}{n}$$

231. *Proposed by Leo Moser, University of Alberta.*

A number is called palindromic if it reads the same forward and backward, when written with the base 10. Find four palindromic primes in arithmetic progression less than 10^6.

232. *Proposed by Leon Bankoff, Los Angeles, California.*

Employing a minimum number of operations with straight-edge and compasses, dissect a square into three segments whose areas are in the ratio 3:4:5.

233. *Proposed by V. Thebault, Tennie, Sarthe, France.*

The points Q, P are the projections upon the sides C and B of a triangle ABC of the feet B', C' of the altitudes BB' and CC'. Show that, (a) PQ is parallel to BC, and the distance between the two parallel lines is equal to S/R, where S is the area and R is the circumradius of ABC. (b) The lines $B'Q$ and $C'P$ are perpendicular to each other and are equally inclined to the bisectors of the angle A.

234. *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Given an m by n rectangular lattice containing mn points, find the total number of (a) squares, (b) rectangles having vertices at the points of the lattice. Consider $m \geq n$.

235. *Proposed by N. Shklov, University of Saskatchewan.*

Find the coordinates of the point P whose distances from each of the points $(1/y_1, y_1)$; $(1/y_2, y_2)$ and $(y_1 y_2 y_3, \frac{1}{y_1 y_2 y_3})$ are equal.

SOLUTIONS

Late Solutions

169. *Ralph Everett, Mississippi Southern College.*

196. *Murray S. Klamkin, Polytechnic Institute of Brooklyn.*

Trigonometric Products

208. [September 1954] *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Evaluate the following reigonomedric expressions without using numerical tables:

$$A = \cos 5^\circ \cos 10^\circ \cos 15^\circ \cdots \cos 75^\circ \cos 80^\circ \cos 85^\circ,$$
$$B = \cos 1^\circ \cos 3^\circ \cos 5^\circ \cdots \cos 85^\circ \cos 87^\circ \cos 89^\circ,$$
$$C = \cos 4^\circ \cos 8^\circ \cos 12^\circ \cdots \cos 80^\circ \cos 84^\circ \cos 88^\circ.$$

I. *Solution by E. P. Starke, Rutgers University.*

$$\sqrt{2}\, A = \cos 5^\circ \cos 10^\circ \cdots \cos 40^\circ \sin 40^\circ \cdots \sin 10^\circ \sin 5^\circ$$
$$= \sin 10^\circ \sin 20^\circ \cdots \sin 80^\circ/2^8,$$

by use of the double-angle formula. A repetition of the same device gives: $2^{12}\sqrt{2}\, A = \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \sqrt{3}\, k/2$, say, where $k = \sin 20^\circ(\sin 40^\circ \sin 80^\circ) = \sin 20^\circ(\sin^2 60^\circ - \sin^2 20^\circ)$ $= \frac{1}{4}(3 \sin 20^\circ - 4 \sin^3 20^\circ) = \frac{1}{4} \sin 60^\circ = \sqrt{3}/8$. So $A = 3 \cdot 2^{-33/2}$.

Similarly $B = \cos 1^\circ \sin 1^\circ \cos 3^\circ \sin 3^\circ \cdots$ $\cos 43^\circ \sin 43^\circ \cos 45^\circ \cdot 2^{22}\sqrt{2}\, B = \sin 2^\circ \sin 6^\circ \sin 10^\circ \ldots \sin 86^\circ = C.$

Let $x = \cos 2^\circ \cos 6^\circ \cos 10^\circ \ldots \cos 86^\circ$.

Then $2^{22} \cdot 2^{22}\sqrt{2}\, B \cdot x = \sin 4^\circ \sin 12^\circ \sin 20^\circ \cdots \sin 172^\circ$

$$= \sin 4^\circ \sin 8^\circ \sin 12^\circ \sin 16^\circ \cdots \sin 88^\circ = x,$$

whence $B = 2^{-89/2}$ and $C = 2^{-22}$.

II. *Solution by H. M. Feldman, Washington University, St. Louis Missouri.*

From the identities

$$x^{2n} - 1 = (x^2 - 1)(x^{2n} + x^{2n-1} + \cdots + 1) = (x^2 - 1)\prod_{k=1}^{n-1}(x^2 - 2x\cos\frac{k\pi}{n} + 1) \quad \text{and}$$

$$x^{2n+1} = (x+1)(x^{2n} - x^{2n-1} + \cdots + 1) = (x+1)\prod_{k=1}^{n}(x^2 - 2x\cos\frac{2k-1}{2n+1} + 1)$$

we get, by letting $x = \pm 1$, the following relations:

$$\prod_1^{n-1} \sin\frac{k\pi}{2n} = \prod_1^{n-1}\cos\frac{k\pi}{n} = 2^{-n+1}\sqrt{n}\ ;$$

$$\prod_1^{n-1} \sin\frac{k\pi}{n}\ 2^{-n+1} n;\ \prod_1^{n}\sin\frac{2k-1}{2(2n+1)}\pi = 2^{-n}$$

$$\text{and} \prod_1^{n}\cos\frac{2k-1}{2(2n+1)} = \prod_1^{n}\sin\frac{2k-1}{2n+1}\pi = 2^{-n}\sqrt{2n+1}.$$

By means of these relations, we find:

$$A = 2^{-17}(3\sqrt{2})$$
$$B = 2^{-45}\sqrt{2}$$
$$C = 2^{-22}$$

Also solved by Leon Bankoff, Los Angeles, California; Kwan Moon (partially), Mississippi State College; George Mott, Republic Aviation Corp., New York; T. F. Mulcrone, St. Charles College, Louisiana; L. A. Ringenberg, Eastern Illinois State College; Chih-yi Wang, University of Minnesota; Hazel S. Wilson, Jacksonville State College, Alabama and the proposer.

A Symmetric Function

209. September 1954 *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn.*

Show that $F(x,y)\ \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{a^{n+1} + y}$ is symmetric in x and y.

Solution by G. Polya, Stanford University.

(1) Heuristic consideration. Symmetry of $F(x,y)$ means that x and y are interchangeable, yet the proposed expression does not render such interchangeability immediately clear. So, we desire another expression for $F(x,y)$ which does render it immediately clear. What kind of expression? As the proposed expression is a power series in x, it is natural to think of a power series in x and y (the Maclaurin expansion in these two variables). How can one obtain it? Expand in powers of y.

(2) Proof. Expanding in geometric series we obtain:

$$F(x,y) = \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{a^{m+1}} \left[1 + \frac{y}{a^{m+1}} \right]^{-1}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{a^{m+1}} \sum_{n=0}^{\infty} \left[- \frac{y}{a^{m+1}} \right]^n$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n} x^m y^n}{a^{(m+1)(n+1)}}$$

in which expression x and y are obviously interchangeable.

(3) Critique. If $|a| \geq 1$, the forgoing transformations are easily justified provided that $|x| < |a|$, $|y| < |a|$. If, however, $0 < |a| < 1$ the proposed series does not behave symmetrically in x and y. In fact, its sum is a regular (analytic) function in a certain neighborhood of the point $x = -a$, $y = a$, whereas it becomes infinite (has a pole) at the point $x = a$, $y = -a$. If $a = 0$, $F(x,y) = y^{-1}(1+x)^{-1}$ and the asymmetry is quite obvious.

Also solved by the proposer.

A Finite Limit.

210. [September 1954] *Proposed by Barney Bissinger, Lebanon Valley College, Pennsylvania.*

A differential equation yielded the solution:

$$y = \frac{g^{c^t} - g}{g^t - 1} \quad \text{for } t > 0.$$

How should the exponents be interpreted to make y finite as $t \to 0$?

Solution by Walter B. Carver, Cornell University. The problem seems to imply that there is more than one correct way to interpret the exponents in the problem. This is not true. The notation g^{c^t} means $g^{(c^t)}$ and not $(g^c)^t$. (The latter form may be more simply expressed as g^{ct}.) The problem becomes trivial or meaningless when $c = 0$ or 1, and when $g = 0$ or 1. Hence we assume $c \neq 0$, $c \neq 1$, $g \neq 0$, $g \neq 1$. Then $g^{c^t} - g$ and $g^t - 1$ both vanish for $t = 0$, and hence

$$\lim_{t\to 0} \frac{g^{c^t} - g}{g^t - 1} = \lim_{t\to 0} \frac{g^{c^t}\cdot\log g\cdot c^t\cdot\log c}{g^t\cdot\log g} = g\cdot\log c$$

hence y remains finite as $t \to 0$ for the only proper interpretation of the exponents.

Also solved by Ben K. Gold, Los Angeles City College; Murray S. Klamkin, Polytechnic Institute of Brooklyn; George Mott, Republic Aviation Corp. and the proposer.

Sums Of Squares

211. [September 1954] *Proposed by J. Lambeck, McGill University and L. Moser, University of Alberta.*

Prove that $\sum_{i=1}^{n^2+n} \{\sqrt{i}\} = 2\sum_{i=1}^{n} i^2$ where $\{x\}$ denotes the integer closest to x.

I. *Solution by L. Carlitz, Duke University.* By the definition of nearest integer, $\{\sqrt{x}\} = m$ if and only if

$$m - 1/2 \le \sqrt{x} \le m + 1/2$$

(actually $\sqrt{x} = m \pm 1/2$ is impossible), which is equivalent to

$$m^2 - m + 1 \le x \le m^2 + m.$$

Thus

$$\sum_{x=1}^{n^2+n}\{\sqrt{x}\} = \sum_{m=1}^{n}\ \sum_{x=m^2-m+1}^{m^2+m} \{\sqrt{x}\}$$

$$= \sum_{m=1}^{n}\ \sum_{x=m^2-m+1}^{m^2+m} m$$

$$= \sum_{m=1}^{n} m(2m) = \sum_{m=1}^{n} m^2$$

II. *Solution by the Proposers.* For $n = 1$ the theorem is obviously true. Proceed by induction over n. For $n^2 + n < x \le (n+1)^2 + (n+1)$ we have $\{\sqrt{x}\} = n + 1$.

Hence

$$\sum_{i=n^2+n+1}^{(n+1)^2+(n+1)} \{\sqrt{i}\} = [(n+1)^2 + (n+1) - (n^2+n)](n+1)$$

$= 2(n+1)^2$ and the induction is complete.

Also solved by Harvey H. Berry, University of Cincinnati; Huseyin Demir, Zonguldak, Turkey; Abraham L. Epstein, Cambridge Research

Center, Massachussets; Murray S. Klamkin, Polytechnic Institute of Brooklyn; Thomas F. Mulcrone, S. J., St Charles College, Louisiana; Lawrence A. Ringenberg, Eastern Illinois State College; Robert E. Shafer, University of California at Berkeley; E. P. Starke, Rutgers University and Chih-yi Wang, University of Minnesota.

A Fantastic Integer

212. [September 1954] *Proposed by F. J. Duarte, Caracas, Venezuela.*

Prove that the number $\dfrac{1 + [1 + (10^{10} - 1)^{99989}]\ (10^{999890} - 1)}{99991}$ is an integer.

Solution by Robert E. Shafer, University of California at Berkeley.

To prove that $\dfrac{1 + [1 + (10^{10} - 1)^{99989}](10^{999890} - 1)}{99991}$ is an integer is equivalent to showing that

$$1 + [1\ (10^{10} - 1)^{99989}]\,[10^{999890} - 1] \equiv 0 \pmod{99991}.$$

We find that $99991 = 10^5 - 9$ is a prime number.

Next $10^{10} - 1 = (10^5 + 1)(10^5 - 1) \equiv 10 \cdot 8 \equiv 80 \pmod{99991}$

and $10^{999890} = ((10^5)^2)^{99989} \equiv (9^2)^{99989} \pmod{9991}$.

Hence by writing $80 = r$ we have

$$1 + \left(1 + \frac{1}{r}\right)\left(\frac{-r}{r+1}\right) \equiv 1 + \left(\frac{r+1}{r}\right)\left(\frac{-r}{r+1}\right)$$

$$\equiv 1 + (-1) \equiv 0 \pmod{9991}$$

which establishes that the number is an integer.

Also solved by Huseyin Demir, Zonguldak, Turkey; E. P. Starke, Rutgers University and the proposer.

A Complete Victory

213. [September 1954] *Proposed by John M. Howell, Los Angeles City College.*

A has a dollars and B has b dollars. They play a series of games in which A has a probability p of winning, probability q of losing and probability t of tying in each game. If they wager one dollar on each game, what is the probability of A winning all of the money?

Solution by the proposer. Let $P(x)$ be the probability of A winning all the money if he starts with x dollars.
$P(x) = pP(x+1) = tP(x) + qP(x-1)$. If $P(x)$ is of the form r^x, then $r^x = pr^{x+1} + tr^x + qr^{x-1}$ or $r = pr^2 + tr + q$, and $pr^2 + (t-1)r + q = 0$ which becomes $pr^2 + (-p-q)r + q = 0$ or $(pr-q)(r-1) = 0$ so that $r = 1$ or $r = p/q$.

Next
$$P(x) = C_1 + C_2(q/p)^x$$
$$P(0) = 0 = C_1 + C_2$$
$$P(a+b) = 1 = C_1 + C_2(q/p)^{a+b}$$
$$C_1 = C_2 = \frac{1}{(q/p)^{a+b} - 1}$$

Then
$$P(x) = \frac{(q/p)^x - 1}{(q/p)^{a+b} - 1} \quad \text{if } p \neq q.$$

Thus the probability of winning starting with a dollars is
$$P(a) = \frac{(q/p)^a - 1}{(q/p)^{a+b} - 1}$$

Also
$$P(x) = \frac{(q/p)^x - 1}{(q/p)^{a+b} - 1} = \frac{\sum_{i=1}^{x-1} (q/p)^i}{\sum_{i=1}^{a+b-1} (q/p)^i}$$

so that if $p = q = \frac{1}{2}$, then $P(x) = \frac{x}{a+b}$ and the probability of winning if A starts with a dollars is $\frac{a}{a+b}$.

Also solved by George Mott, Republic Aviation Company.

A Circle of Similitude

214. [September 1954] *Proposed by N. A. Court, University of Oklahoma.*

The radical circle of two given orthogonal circles is the circle of similitude of the two circles of antisimilitude of the given circles.

Solution by the proposer. Let (A), (B) be two orthogonal circles. centers A, B. The circles (X), (X') coaxal with (A), (B) and having for centers the centers of similitude X, X' of (A), (B) are the circles of antisimilitude of the latter circles (see, for inst., Nathan Altshiller-Court, College Geometry, sec. ed., p. 214, art. 482. New York, 1952).

The radical circle (AB) of the two orthogonal circles (A), (B) is coaxal with those circles and has for diameter their line of centers AB (ibid. p. 215, art. 487).

The circle (AB) is thus coaxal with the circles (X), (X') and the two points A,B in which (AB) meets their line of centers XX' separate their centers X, X' harmonically (ibid. p. 185, art. 395), hence the proposition (ibid. p. 214, art. 481).

Inverse Functions

187. [January 1954] *Proposed by B. K. Gold's Calculus Class, Los Angeles City College.*

A student differentiated $f(x) = \text{arc sin } \dfrac{x^2 - a^2}{x^2 + a^2}$ and $g(x) = 2\,\text{arc sin } x/\sqrt{x^2 + a^2}$ and noted that their derivatives were equal. He reasoned that the anti-derivatives of $f'(x)$ and $g'(x)$ must differ only by an additive constant. Show that this is true.

Editor's note. A number of comments have been received concerning this problem since a solution was published in the September 1954 issue.

Comments by Walter B. Carver, Cornell University

Something more than a purely formal solution to problem 187 might be desirable. It seems to me that the solution appearing in the September-October issue does not reveal the true significance of the problem.

The student who differentiated the functions $f(x)$ and $g(x)$ and "noted that their derivatives were equal" made a mistake to begin with. The derivatives of these two functions are equal *only for positive values of x*. The derivative $g'(x)$ is $2a/(x^2 + a^2)$ for all x, while $f'(x)$ is $2a/(x^2+a^2)$ for positive and $-2a/(x^2+a^2)$ for negative x.

One can not speak of the derivatives of the inverse sine and inverse cosine functions until there is an agreement on some convention which will make these functions single-valued. All elementary calculus texts do this by specifying that

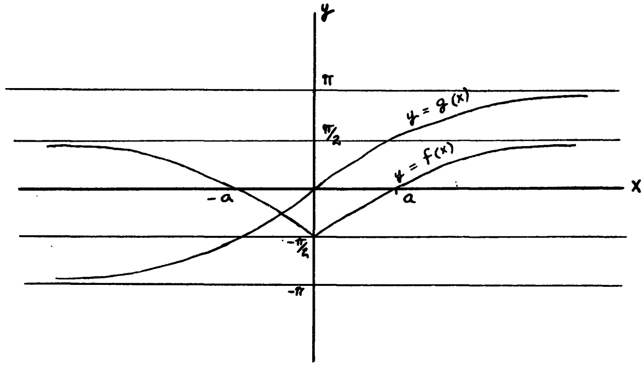
$$-\pi/2 \leq \text{arc sin } x \leq \pi/2 \text{ and } 0 \leq \text{arc cos } x \leq \pi,$$

and the formulas given for differentiating these functions are correct only if such restrictions are observed. The relation

$$\text{arc sin } A - \text{arc cos } A = \text{arc sin } (2A^2 - 1)$$

with which the published solution begins does not hold for negative A, as may be readily seen by giving A such values as $-\frac{1}{2}$ or $-1/\sqrt{2}$. And since A has the same sign as x, the conclusions reached in the solution are not valid for negative values of x.

The graphs of the two functions $f(x)$ and $g(x)$ are as follows:



QUICKIEES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 134. Henry starts a trip into the country between 8:00 AM and 9:00 AM when the hands of the clock are together. He arrives at his destination between 2:00 PM and 3:00 PM when the hands of the clock are exactly 180 degrees apart. How long did he travel? [*Submitted by Charles Salkind*].

Q 135. How many different scores is it possible to make in m throws of n dice? [*Submitted by Monte Dernham*].

Q 136. Let $[x]$ denote as usual the greatest integer less than or equal to x, and let (x) denote the largest integer nearest to x Express (x) as a function of $[x]$. [*Submitted by M. S. Klamkin*] .

Q 137. The reciprocal of a number was discovered to have exactly the same digits as the number itself. What are all such numbers? [*Submitted by Glenn D. James*].

Q 138. Find the sum $S_m = 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!$ [*Submitted by Huseyin Demir*].

Q 139. What is the smallest number of balance weights needed to weigh every integral weight from 1 to 121 pounds? [*Submitted by M.S.Klamkin*]·

ANSWERS

A 134. The trip took six hours. Picture a clock with two hour-hands, one set at 8 and the other at 2, six hours apart. For any given number of minutes m the two hour hands advance the same number of degrees so they remain 180 degrees apart. This is true when m is such that the minute hand coincides in position with the first hour hand. This reasoning can be generalized to a trip of any time less than or equal to 12 hours and starting at any point on the clock.

A 135. The maximum score for any single throw is $6n$, the minimum n, For m throws these would be $6mn$ and mn respectively. But every intervening number may be scored. Hence, there are in all $(6mn + 1) - mn = 5mn + 1$ different scores.

A 136. By plotting $y = [x]$ and $y = (x)$ it is readily seen that $(x) = [x - \frac{1}{2}] + 1 = [x + \frac{1}{2}]$.

A 137. Since the negative mantissa must equal the positive mantissa, all such numbers are of the form $10^{n/2}$ where n is an integer. In particular the solutions are the trivial value 1 and the number $\sqrt{10}$.

A 138.

$$S = \sum_{p=1}^{n} p\cdot p! = \sum_{p=1}^{n} (p + 1 - 1)\cdot p!$$

$$= \sum_{p=1}^{n} (p + 1)! - \sum_{p=1}^{n} p!$$

$$= (n + 1)! - 1$$

A 139. If the weights can be placed on only one side of the balance, then the weights needed are 1,2,4,8,16,32,64. If they can be placed on both sides then we need only 1,3,9,27,81.

QUICKENING THE QUICKIES

Q 119. [September 1954] To prove $(21/20)^{100} > 100$ we may write $100 \ln (1+1/20) > \ln 100 = 2(2.303)$ or $100 \ln (1+1/20 - 1/(2\cdot 20^2)) > 4.606$

Here the inequality is proven with only two terms of the expansion and the use of the well known constant $\ln 10 = 2.303$, whereas the previous proof required five terms of the expansion. [*Submitted by M. S. Klamkin.*]